technische universität
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Computational Intelligence

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- Fuzzy sets
  - Axioms of fuzzy complement, t- and s-norms
  - Generators
  - Dual tripels

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

  with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

Definition

A function c: [0,1] → [0,1] is a **fuzzy complement** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall a, b \in [0,1]: a \leq b \Rightarrow c(a) \geq c(b)$.   monotone decreasing

"nice to have":

(A3) $c(\cdot)$ is continuous.

(A4) $\forall a \in [0,1]: c(c(a)) = a$ involutive

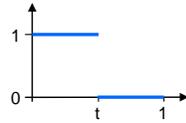**Examples:**

a) standard fuzzy complement $c(a) = 1 - a$

   ad (A1): $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$ ad (A3): ☑
   ad (A2): $c'(a) = -1 < 0$ (monotone decreasing)   ad (A4): $1 - (1 - a) = a$

b) $c(a) = \begin{cases} 1 & \text{if } a \le t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$
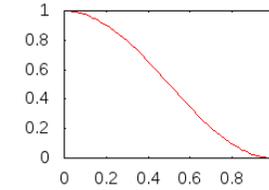
ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t   ☑

ad (A3): not valid → discontinuity at t

ad (A4): not valid → counter example

$c(c(¼)) = c(1) = 0 \ne ¼$ for t = ½

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c) $c(a) = \dfrac{1 + \cos(\pi a)}{2}$
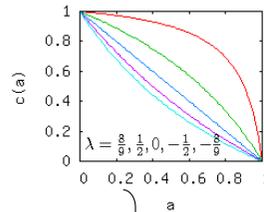
ad (A1): $c(0) = 1$ and $c(1) = 0$ ☑

ad (A2): $c'(a) = -½ \pi \sin(\pi a) < 0$ since $\sin(\pi a) > 0$ for $a \in (0,1)$

ad (A3): is continuous as a composition of continuous functions

ad (A4): not valid → counter example

$c\left(c\left(\frac{1}{3}\right)\right) = c\left(\frac{3}{4}\right) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) \ne \frac{1}{3}$
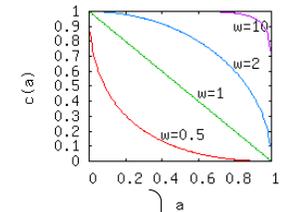
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d) $c(a) = \dfrac{1-a}{1+\lambda a}$ for $\lambda > -1$ **Sugeno class**

$\lambda = \frac{8}{9}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{8}{9}$

ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c(a) \ge c(b) \Leftrightarrow \frac{1-a}{1+\lambda a} \ge \frac{1-b}{1+\lambda b} \Leftrightarrow$

$(1-a)(1+\lambda b) \ge (1-b)(1+\lambda a) \Leftrightarrow$ ☑

$b(\lambda+1) \ge a(\lambda+1) \Leftrightarrow b \ge a$

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left(\frac{1-a}{1+\lambda a}\right) = \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda\frac{1-a}{1+\lambda a}} = \frac{a(\lambda+1)}{\lambda+1} = a$ ☑

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e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$ **Yager class**

$w=10$, $w=2$, $w=1$, $w=0.5$

ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \ge (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \ge 1 - b^w \Leftrightarrow$ ☑
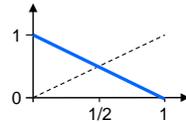
$a^w \le b^w \Leftrightarrow a \le b$

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left((1-a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1-a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$ ☑
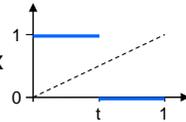
$= (1-(1-a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$

Theorem

If function c:[0,1] → [0,1] satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a* with c(a*) = a*.

Proof:

one fixed point → see example (a) → intersection with bisectrix

no fixed point → see example (b) → no intersection with bisectrix

assume ∃ n > 1 fixed points, for example a* and b* with a* < b*

⇒ c(a*) = a* and c(b*) = b*   (fixed points)

⇒ c(a*) < c(b*) with a* < b* impossible if c(·) is monotone decreasing

⇒ contradiction to axiom (A2)   ∎

Theorem

If function c:[0,1] → [0,1] satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a* with c(a*) = a*.

Proof:

Intermediate value theorem →

If c(·) continuous (A3) and c(0) ≥ c(1) (A1/A2)

then ∀ v ∈ [c(1), c(0)] = [0,1]: ∃ a ∈ [0,1]: c(a) = v.

⇒ there must be an intersection with bisectrix

⇒ a fixed point exists and by previous theorem there are no other fixed points!   ∎

Examples:

(a) $c(a) = 1 - a$ ⇒ $a = 1 - a$ ⇒ $a^* = ½$

(b) $c(a) = (1 - a^w)^{1/w}$ ⇒ $a = (1 - a^w)^{1/w}$ ⇒ $a^* = (½)^{1/w}$

Theorem

c: [0,1] → [0,1] is involutive fuzzy complement iff

∃continuous function g: [0,1] → ℝ with

• g(0) = 0

• strictly monotone increasing

• ∀ a ∈ [0,1]: c(a) = g^(-1)(g(1) – g(a)).   ∎

defines an **increasing generator**

g^(-1)(x) pseudo-inverse

Examples

a) g(x) = x ⇒ g^-1(x) = x ⇒ c(a) = 1 – a   (Standard)

b) g(x) = x^w ⇒ g^-1(x) = x^{1/w} ⇒ c(a) = (1 – a^w)^{1/w}   (Yager class, w > 0)

c) g(x) = log(x+1) ⇒ g^-1(x) = e^x – 1 ⇒ c(a) = exp(log(2) – log(a+1)) – 1

$$= \frac{1-a}{1+a}$$ (Sugeno class. λ = 1)

Examples

d) $g(a) = \frac{1}{\lambda} \log_e(1 + \lambda a)$ for $\lambda > -1$

• $g(0) = \log_e(1) = 0$

• strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda a} > 0$ for $a \in [0,1]$

• inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda a) - 1}{\lambda}$, thus

$$
\begin{aligned}
c(a) &= g^{-1}\left(\frac{\log(1+\lambda)}{\lambda} - \frac{\log(1+\lambda a)}{\lambda} \right) \\
&= \frac{\exp(\log(1+\lambda) - \log(1+\lambda a)) - 1}{\lambda} \\
&= \frac{1}{\lambda}\left(\frac{1+\lambda}{1+\lambda a} - 1 \right) = \frac{1-a}{1+\lambda a} \quad \text{(Sugeno Complement)}
\end{aligned}
$$

Theorem

c: [0,1] → [0,1] is involutive fuzzy complement iff

∃continuous function f: [0,1] → ℝ with

- f(1) = 0
- strictly monotone decreasing
- ∀ a ∈ [0,1]: c(a) = f$^{(-1)}$(f(0) − f(a)). ∎

} defines a **decreasing generator**

f$^{(-1)}$(x) pseudo-inverse

Examples

a) f(x) = $k - k \cdot x$ $(k > 0)$ f$^{(-1)}$(x) = 1 − x/k c(a) = $1 - \frac{k-(k-ka)}{k} = 1 - a$

b) f(x) = 1 − x^w f$^{(-1)}$(x) = (1 − x)$^{1/w}$ c(a) = f^{-1}(a^w) = (1 − a^w)$^{1/w}$ (Yager)

Definition

A function t:[0,1] x [0,1] → [0,1] is a **fuzzy intersection** or **t-norm** iff ∀a,b,d ∈ [0,1]

(A1) t(a, 1) = a   (boundary condition)

(A2) b ≤ d ⇒ t(a, b) ≤ t(a, d)   (monotonicity)

(A3) t(a,b) = t(b, a)   (commutative)

(A4) t(a, t(b, d)) = t(t(a, b), d)   (associative) ∎

"nice to have"

(A5) t(a, b) is continuous   (continuity)

(A6) t(a, a) < a   for 0 < a < 1   (subidempotent)

(A7) a$_1$ < a$_2$ and b$_1$ ≤ b$_2$ ⇒ t(a$_1$, b$_1$) < t(a$_2$, b$_2$)   (strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

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Theorem:

The only idempotent t-norm is the standard fuzzy intersection.

Proof:

Assume there exists a t-norm with t(a,a) = a for all a ∈ [0,1].

- If 0 ≤ a ≤ b ≤ 1 then

  a = t(a,a) ≤ t(a,b) ≤ t(a, 1) = a
      ↑   ↑   ↑   ↑
  by assumption   by monotonicity   by boundary condition

  and hence t(a,b) = a.

- If 0 ≤ b ≤ a ≤ 1 then

  b = t(b,b) ≤ t(b,a) ≤ t(b, 1) = b
      ↑   ↑   ↑   ↑
  by assumption   by monotonicity   by boundary condition

  and hence t(a,b) = t(b,a) = b.
            ↑
        by commutativity

} t(a,b) = min(a,b) is the only possible solution!

q.e.d.

Examples:

Name	Function
(a) Standard	t(a, b) = min { a, b }
(b) Algebraic Product	t(a, b) = a · b
(c) Bounded Difference	t(a, b) = max { 0, a + b − 1 }
(d) Drastic Product	t(a, b) = { a if b = 1 ; b if a = 1 ; 0 otherwise

(a)   (b)

(c)   (d)
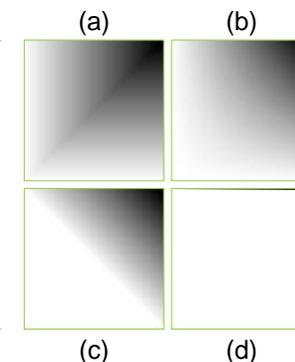
Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): t(a, 1) = a · 1 = a   ☑   ad (A3): t(a, b) = a · b = b · a = t(b, a)   ☑

ad (A2): a · b ≤ a · d ⇔ b ≤ d   ☑   ad (A4): a · (b · d) = (a · b) · d   ☑

Theorem

Function t: [0,1] x [0,1] → [0,1] is a t-norm ⇔

∃decreasing generator f:[0,1] → ℝ with t(a, b) = f^{-1}(min{ f(0), f(a) + f(b) }).   ∎

Example:

f(x) = 1/x − 1 is decreasing generator since

- f(x) is continuous   ☑
- f(1) = 1/1 − 1 = 0   ☑
- f'(x) = $-1/x^2$ < 0 (monotone decreasing)   ☑

inverse function is $f^{-1}(x) = \dfrac{1}{x+1}$;   f(0) = ∞ ⇒ min{ f(0), f(a) + f(b) } = f(a) + f(b)

⇒ t(a, b) = $f^{-1}\left(\dfrac{1}{a} + \dfrac{1}{b} - 2\right)$ = $\dfrac{1}{\frac{1}{a}+\frac{1}{b}-1}$ = $\dfrac{ab}{a+b-a\,b}$

Definition

A function s:[0,1] x [0,1] → [0,1] is a ***fuzzy union*** or ***s-norm*** iff ∀a,b,d ∈ [0,1]

(A1) s(a, 0) = a   (boundary condition)

(A2) b ≤ d ⇒ s(a, b) ≤ s(a, d)   (monotonicity)

(A3) s(a, b) = s(b, a)   (commutative)

(A4) s(a, s(b, d)) = s(s(a, b), d)   (associative)   ∎

"nice to have"

(A5) s(a, b) is continuous   (continuity)

(A6) s(a, a) > a   for 0 < a < 1   (superidempotent)

(A7) a_1 < a_2 and b_1 ≤ b_2 ⇒ s(a_1, b_1) < s(a_2, b_2)   (strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Examples:

Name	Function
Standard	s(a, b) = max { a, b }
Algebraic Sum	s(a, b) = a + b − a · b
Bounded Sum	s(a, b) = min { 1, a + b }
Drastic Union	s(a, b) = $\begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$

(a)   (b)

(c)   (d)

Is algebraic sum a t-norm? Check the 4 axioms!

ad (A1): s(a, 0) = a + 0 − a · 0 = a   ☑

ad (A2): a + b − a · b ≤ a + d − a · d ⇔ b (1 − a) ≤ d (1 − a) ⇔ b ≤ d   ☑   ad (A4): ☑

ad (A3): ☑

Theorem

Function s: [0,1] x [0,1] → [0,1] is a s-norm ⇔

∃increasing generator g:[0,1] → ℝ with s(a, b) = g^{-1}(min{ g(1), g(a) + g(b) }). ∎

Example:

g(x) = −log(1 − x) is increasing generator since

- g(x) is continuous   ☑
- g(0) = −log(1 − 0) = 0   ☑
- g'(x) = 1/(1 − x) > 0 (monotone increasing) ☑

inverse function is $g^{-1}(x)$ = 1 − exp(−x) ; g(1) = ∞ ⇒ min{g(1), g(a) + g(b)} = g(a) + g(b)

⇒ s(a, b) = $g^{-1}(-\log(1-a) - \log(1-b))$

= $1 - \exp(\log(1-a) + \log(1-b))$

= $1 - (1-a)(1-b)$ = $a + b - a\,b$ (algebraic sum)

Background from classical set theory:

$\cap$ and $\cup$ operations are dual w.r.t. complement since they obey DeMorgan's laws

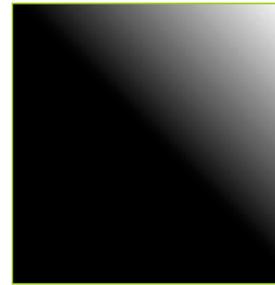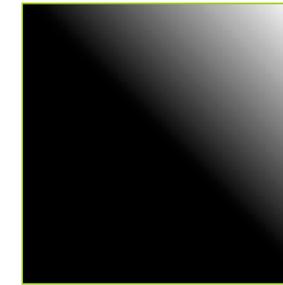| **Definition**	**Definition**
A pair of t-norm $t(\cdot, \cdot)$ and s-norm $s(\cdot, \cdot)$ is said to be **dual with regard to the fuzzy complement** $c(\cdot)$ iff • $c(\,t(a, b)\,) = s(\,c(a), c(b)\,)$ • $c(\,s(a, b)\,) = t(\,c(a), c(b)\,)$ for all $a, b \in [0,1]$. ∎	Let (c, s, t) be a tripel of fuzzy complement $c(\cdot)$, s- and t-norm. If t and s are dual to c then the tripel (c,s,t) is called a **dual tripel**. ∎

Examples of dual tripels

t-norm	s-norm	complement
min { a, b }	max { a, b }	1 – a
a · b	a + b – a · b	1 – a
max { 0, a + b – 1 }	min { 1, a + b }	1 – a

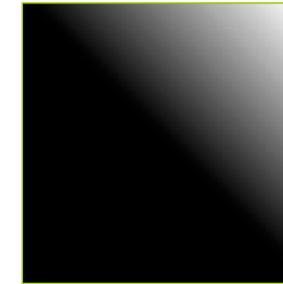c(t(a, b))   s(c(a), c(b))

Dual Triple:

- bounded difference

- bounded sum

- standard complement

$\Rightarrow$ left image = right image

Non-Dual Triple:

- algebraic product

- bounded sum

- standard complement

$\Rightarrow$ left image ≠ right image

Why are dual triples so important?

$\Rightarrow$ allow equivalence transformations of fuzzy set expressions

$\Rightarrow$ required to transform into some equivalent normal form (standardized input)

$\Rightarrow$ e.g. two stages: intersection of unions $\quad \bigcap_{i=1}^{n} (A_i \cup B_i)$

or union of intersections $\quad \bigcup_{i=1}^{n} (A_i \cap B_i)$

Example:

$A \cup (B \cap (C \cap D)^c) =$	← not in normal form
$A \cup (B \cap (C^c \cup D^c)) =$	← equivalent if DeMorgan's law valid (dual triples!)
$A \cup (B \cap C^c) \cup (B \cap D^c)$	← equivalent (distributive lattice!)