Computational Intelligence

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Prof. Dr. Günter Rudolph

Lehrstuhl für Algorithm Engineering (LS 11)

Fakultät für Informatik

TU Dortmund

- Fuzzy sets

  - Axioms of fuzzy complement, t- and s-norms

  - Generators

  - Dual tripels

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$

- $(A \cap B)(x) = \min \{ A(x), B(x) \}$

- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

  with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.

- Creation via generators.

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Definition

A function c: $[0,1] \rightarrow [0,1]$ is a ***fuzzy complement*** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall\, a, b \in [0,1]: a \leq b \Rightarrow c(a) \geq c(b)$.   monotone decreasing

"nice to have":

(A3) $c(\cdot)$ is continuous.

(A4) $\forall\, a \in [0,1]: c(c(a)) = a$ involutive

Examples:

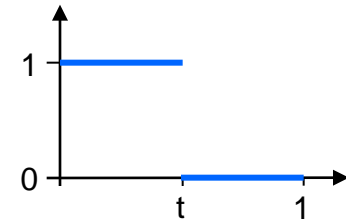a) standard fuzzy complement $c(a) = 1 - a$

ad (A1): $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$ ad (A3): ☑

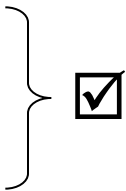ad (A2): $c'(a) = -1 < 0$ (monotone decreasing)   ad (A4): $1 - (1 - a) = a$

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b) $c(a) = \begin{cases} 1 & \text{if } a \le t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$

ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

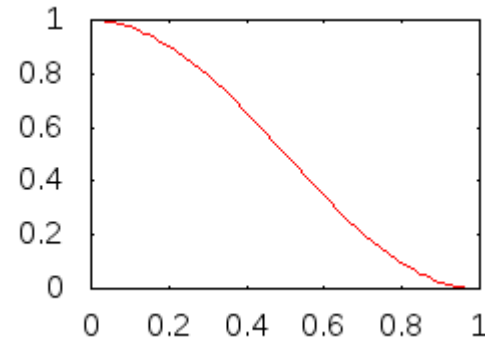ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t ☑

ad (A3): not valid → discontinuity at t

ad (A4): not valid → counter example

   $c(c(¼)) = c(1) = 0 \ne ¼$ for $t = ½$

c) $c(a) = \dfrac{1 + \cos(\pi a)}{2}$

ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c'(a) = -\tfrac{1}{2}\,\pi\,\sin(\pi a) < 0$ since $\sin(\pi a) > 0$ for $a \in (0,1)$
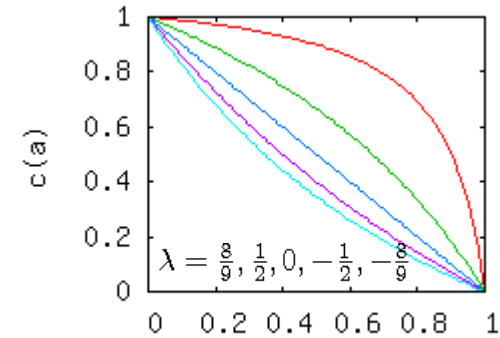
☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): not valid → counter example

$$c\left(c\left(\tfrac{1}{3}\right)\right) = c\left(\tfrac{3}{4}\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{\sqrt{2}}\right) \neq \tfrac{1}{3}$$

d) c(a) = $\dfrac{1-a}{1+\lambda\,a}$　for　$\lambda > -1$　　　**Sugeno class**

ad (A1):　c(0) = 1 and　c(1) = 0

ad (A2):　$c(a) \geq c(b) \;\Leftrightarrow\; \dfrac{1-a}{1+\lambda\,a} \geq \dfrac{1-b}{1+\lambda\,b} \;\Leftrightarrow\;$

$(1-a)\,(1+\lambda\,b) \geq (1-b)\,(1+\lambda\,a) \;\Leftrightarrow\;$

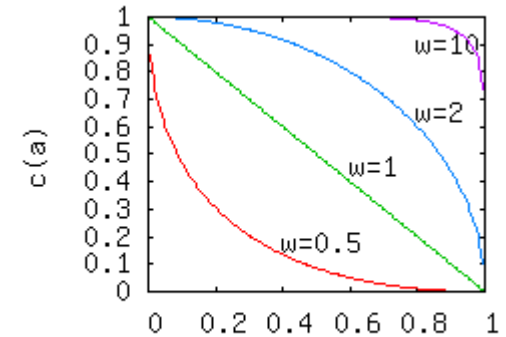$b\,(\lambda+1) \geq a\,(\lambda+1) \;\Leftrightarrow\; b \geq a$

☑

ad (A3):　is continuous as a composition of continuous functions

ad (A4):　$c(c(a)) \;=\; c\left(\dfrac{1-a}{1+\lambda\,a}\right) \;=\; \dfrac{1-\frac{1-a}{1+\lambda\,a}}{1+\lambda\,\frac{1-a}{1+\lambda\,a}} \;=\; \dfrac{a\,(\lambda+1)}{\lambda+1} \;=\; a$

☑

e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$ **Yager class**

ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \geq (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \geq 1 - b^w \Leftrightarrow$

$a^w \leq b^w \Leftrightarrow a \leq b$

☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left((1 - a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1 - a^w)^{\frac{1}{w}}\right]^{w}\right)^{\frac{1}{w}}$

$= (1 - (1 - a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$
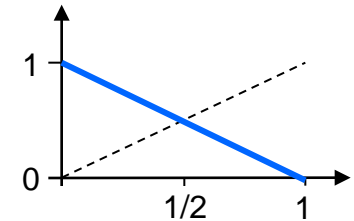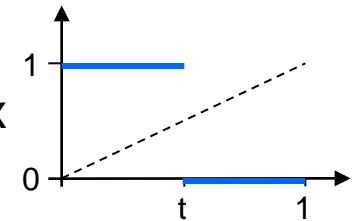
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Theorem

If function $c:[0,1] \to [0,1]$ satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a^* with $c(a^*) = a^*$.

Proof:

one fixed point $\to$ see example (a) $\to$ intersection with bisectrix

no fixed point $\to$ see example (b) $\to$ no intersection with bisectrix

assume $\exists\, n > 1$ fixed points, for example a^* and b^* with $a^* < b^*$

$\Rightarrow c(a^*) = a^*$ and $c(b^*) = b^*$ (fixed points)

$\Rightarrow c(a^*) < c(b^*)$ with $a^* < b^*$ impossible if $c(\cdot)$ is monotone decreasing

$\Rightarrow$ contradiction to axiom (A2) ∎

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Theorem

If function c:[0,1] → [0,1] satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a* with c(a*) = a*.

Proof:

Intermediate value theorem →

If c(·) continuous (A3) and c(0) ≥ c(1) (A1/A2)

then $\forall$ v $\in$ [c(1), c(0)] = [0,1]: $\exists$ a $\in$ [0,1]: c(a) = v.

$\Rightarrow$ there must be an intersection with bisectrix

$\Rightarrow$ a fixed point exists and by previous theorem there are no other fixed points! ∎

Examples:

(a) c(a) = 1 − a $\Rightarrow$ a = 1 − a $\Rightarrow$ a* = ½

(b) c(a) = $(1 - a^w)^{1/w}$ $\Rightarrow$ a = $(1 - a^w)^{1/w}$ $\Rightarrow$ a* = $(½)^{1/w}$

Theorem

c: $[0,1] \rightarrow [0,1]$ is involutive fuzzy complement iff

$\exists$ continuous function g: $[0,1] \rightarrow \mathbb{R}$ with

- $g(0) = 0$

- strictly monotone increasing

- $\forall a \in [0,1]: c(a) = g^{(-1)}(g(1) - g(a))$.   ■

defines an
increasing generator

$g^{(-1)}(x)$ pseudo-inverse

Examples

a) $g(x) = x$ $\Rightarrow g^{-1}(x) = x$ $\Rightarrow c(a) = 1 - a$ (Standard)

b) $g(x) = x^w$ $\Rightarrow g^{-1}(x) = x^{1/w}$ $\Rightarrow c(a) = (1 - a^w)^{1/w}$ (Yager class, w > 0)

c) $g(x) = \log(x+1) \Rightarrow g^{-1}(x) = e^x - 1 \Rightarrow c(a) = \exp(\log(2) - \log(a+1)) - 1$

$$= \frac{1 - a}{1 + a}$$ (Sugeno class. $\lambda = 1$)

Examples

d) $g(a) = \frac{1}{\lambda}\log_e(1 + \lambda\, a)$ for $\lambda > -1$

- $g(0) = \log_e(1) = 0$

- strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda\, a} > 0$ for $a \in [0, 1]$

- inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda\, a) - 1}{\lambda}$, thus

$$
\begin{aligned}
c(a) &= g^{-1}\left(\frac{\log(1 + \lambda)}{\lambda} - \frac{\log(1 + \lambda\, a)}{\lambda}\right) \\[2mm]
&= \frac{\exp(\log(1 + \lambda) - \log(1 + \lambda\, a)) - 1}{\lambda} \\[2mm]
&= \frac{1}{\lambda}\left(\frac{1 + \lambda}{1 + \lambda\, a} - 1\right) = \frac{1 - a}{1 + \lambda\, a} \quad \text{(Sugeno Complement)}
\end{aligned}
$$

Theorem

c: $[0,1] \rightarrow [0,1]$ is involutive fuzzy complement iff

$\exists$ continuous function f: $[0,1] \rightarrow \mathbb{R}$ with

• $f(1) = 0$

• strictly monotone decreasing

• $\forall\, a \in [0,1]: c(a) = f^{(-1)}(\, f(0) - f(a)\,)$.   ∎

defines a
decreasing generator

$f^{(-1)}(x)$ pseudo-inverse

Examples

a) $f(x) = k - k \cdot x$ $(k > 0)$ $\quad f^{(-1)}(x) = 1 - x/k \qquad c(a) = 1 - \dfrac{k - (k - ka)}{k} = 1 - a$

b) $f(x) = 1 - x^w \qquad\qquad f^{(-1)}(x) = (1 - x)^{1/´w} \quad c(a) = f^{-1}(a^w) = (1 - a^w)^{1/w}$ (Yager)

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Definition

A function $t:[0,1] \times [0,1] \to [0,1]$ is a ***fuzzy intersection*** or ***t-norm*** iff

(A1) $t(a, 1) = a$

(A2) $b \leq d \Rightarrow t(a, b) \leq t(a, d)$ (monotonicity)

(A3) $t(a,b) = t(b, a)$ (commutative)

(A4) $t(a, t(b, d)) = t(t(a, b), d)$ (associative)   ■

"nice to have"

(A5) $t(a, b)$ is continuous   (continuity)

(A6) $t(a, a) < a$ (subidempotent)

(A7) $a_1 \leq a_2$ and $b_1 \leq b_2 \Rightarrow t(a_1, b_1) \leq t(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

Examples:

Name	Function
Standard	$t(a, b) = \min \{ a, b \}$
Algebraic Product	$t(a, b) = a \cdot b$
Bounded Difference	$t(a, b) = \max \{ 0, a + b - 1 \}$
Drastic Product	$t(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$

Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): $t(a, 1) = a \cdot 1 = a$ ☑   ad (A3): $t(a, b) = a \cdot b = b \cdot a = t(b, a)$ ☑

ad (A2): $a \cdot b \leq a \cdot d \Leftrightarrow b \leq d$ ☑   ad (A4): $a \cdot (b \cdot d) = (a \cdot b) \cdot d$ ☑

Theorem

Function t: $[0,1] \times [0,1] \to [0,1]$ is a t-norm $\Leftrightarrow$

$\exists$ decreasing generator f: $[0,1] \to \mathbb{R}$ with $t(a, b) = f^{(-1)}(f(a) + f(b))$. ■

Example:

$f(x) = 1/x - 1$ is decreasing generator since

- $f(x)$ is continuous ☑

- $f(1) = 1/1 - 1 = 0$ ☑

- $f'(x) = -1/x^2 < 0$ (monotone decreasing) ☑

inverse function is $f^{-1}(x) = \dfrac{1}{x+1}$

$$\Rightarrow t(a, b) = f^{-1}\left(\frac{1}{a} + \frac{1}{b} - 2\right) = \frac{1}{\frac{1}{a}+\frac{1}{b}-1} = \frac{ab}{a+b-a\,b}$$

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Definition

A function $s:[0,1] \times [0,1] \to [0,1]$ is a *fuzzy union* or *s-norm* or *t-conorm* iff

(A1) $s(a, 0) = a$

(A2) $b \leq d \Rightarrow s(a, b) \leq s(a, d)$ (monotonicity)

(A3) $s(a, b) = s(b, a)$ (commutative)

(A4) $s(a, s(b, d)) = s(s(a, b), d)$ (associative)   ■

"nice to have"

(A5) $s(a, b)$ is continuous   (continuity)

(A6) $s(a, a) > a$ (superidempotent)

(A7) $a_1 \leq a_2$ and $b_1 \leq b_2 \Rightarrow s(a_1, b_1) \leq s(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Examples:

Name	Function
Standard	$s(a, b) = \max \{ a, b \}$
Algebraic Sum	$s(a, b) = a + b - a \cdot b$
Bounded Sum	$s(a, b) = \min \{ 1, a + b \}$
Drastic Union	$s(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$

Is algebraic sum a t-norm? Check the 4 axioms!

ad (A1): $s(a, 0) = a + 0 - a \cdot 0 = a$ ☑

ad (A2): $a + b - a \cdot b \leq a + d - a \cdot d \Leftrightarrow b (1 - a) \leq d (1 - a) \Leftrightarrow b \leq d$ ☑

ad (A3): ☑

ad (A4): ☑

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Theorem

Function s: $[0,1] \times [0,1] \to [0,1]$ is a s-norm $\Leftrightarrow$

$\exists$ increasing generator g:$[0,1] \to \mathbb{R}$ with s(a, b) = g$^{(-1)}$(g(a) + g(b)).   ■

Example:

g(x) = –log(1 – a) is increasing generator since

- g(x) is continuous   ☑

- g(0) = –log(1 – 0) = 0   ☑

- g'(x) = 1/(1-a) > 0 (monotone increasing)   ☑

inverse function is g^{-1}(x) = 1 – exp(–a)

$$\Rightarrow s(a, b) = g^{-1}(-\log(1-a) - \log(1-b))$$
$$= 1 - \exp(\log(1-a) + \log(1-b))$$
$$= 1 - (1-a)(1-b) = a + b - a\,b \quad \text{(algebraic sum)}$$

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Background from classical set theory:

∩ and ∪ operations are dual w.r.t. complement since they obey DeMorgan's laws

Definition
A pair of t-norm t(·, ·) and s-norm s(·, ·) is said to be ***dual with regard to the fuzzy complement*** c(·) iff • c(t(a, b)) = s(c(a), c(b)) • c(s(a, b)) = t(c(a), c(b)) for all a, b ∈ [0,1]. ∎

Definition
Let (c, s, t) be a tripel of fuzzy complement c(·), s- and t-norm. If t and s are dual to c then the tripel (c,s, t) is called a ***dual tripel***. ∎

Examples of dual tripels

t-norm	s-norm	complement
min { a, b }	max { a, b }	$1 - a$
$a \cdot b$	$a + b - a \cdot b$	$1 - a$
max { 0, a + b − 1 }	min { 1, a + b }	$1 - a$