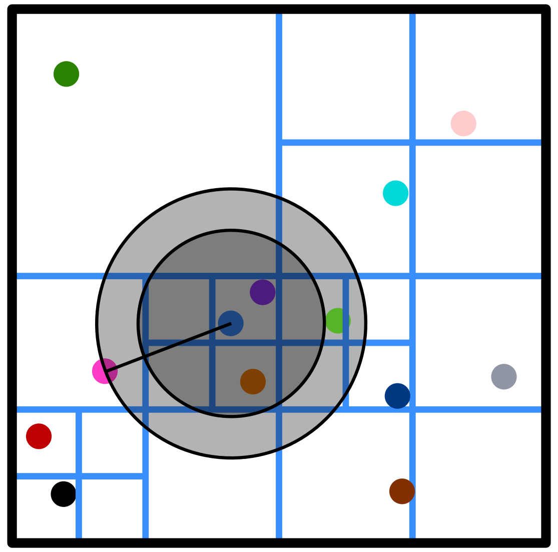
Semi-separated pair decomposition &

low-quality approximate nearest neighbors

Motivation: low-quality approximate nearest neighbors

1. (Compressed) quadtree returns $(1+\varepsilon)$-approximate nearest neighbor of q in time $O(1/\varepsilon^d + \log(1/r))$ time, where $r = d(q, nn(q))$



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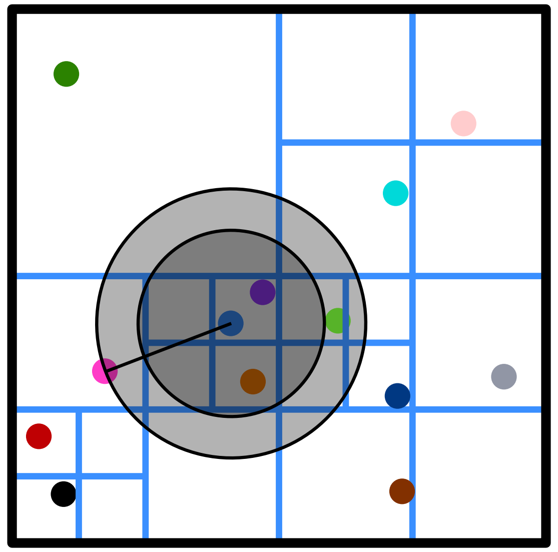

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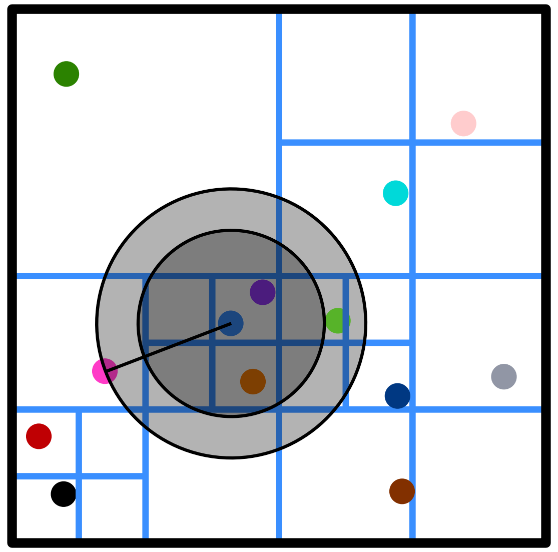

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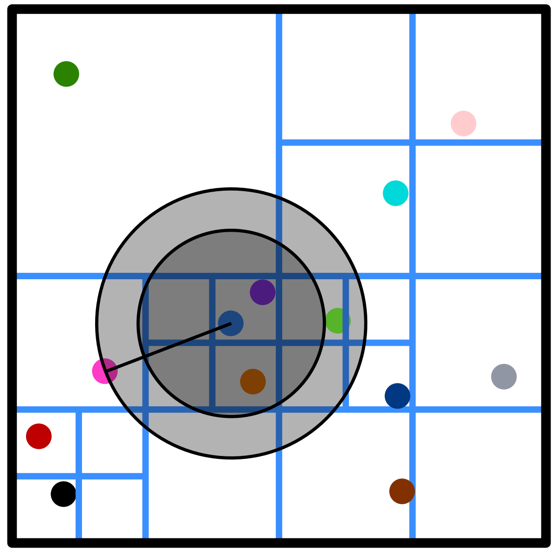

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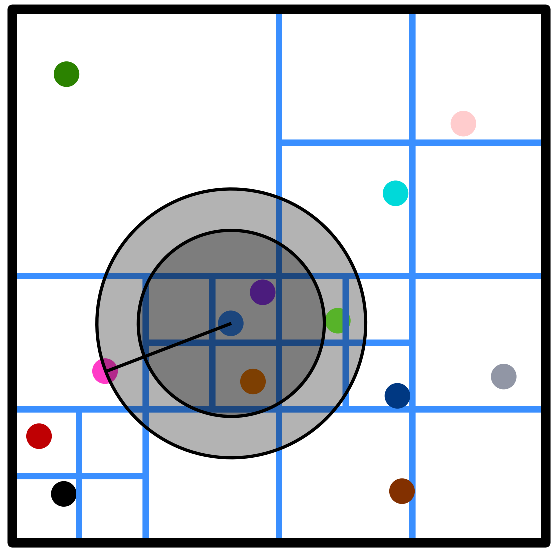

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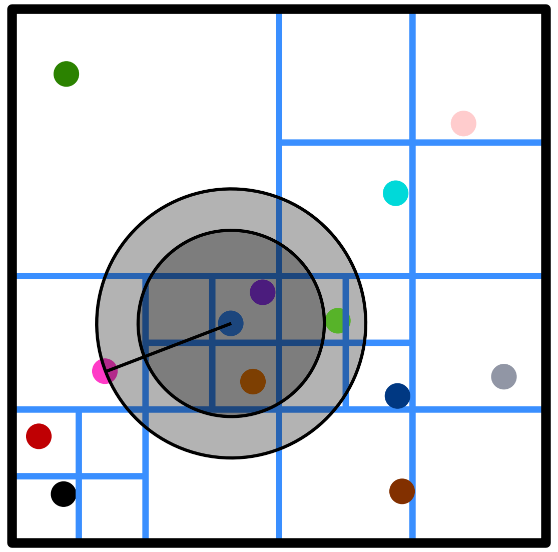

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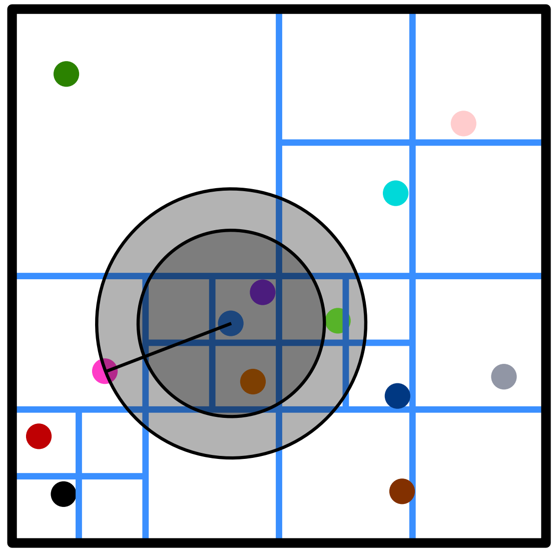

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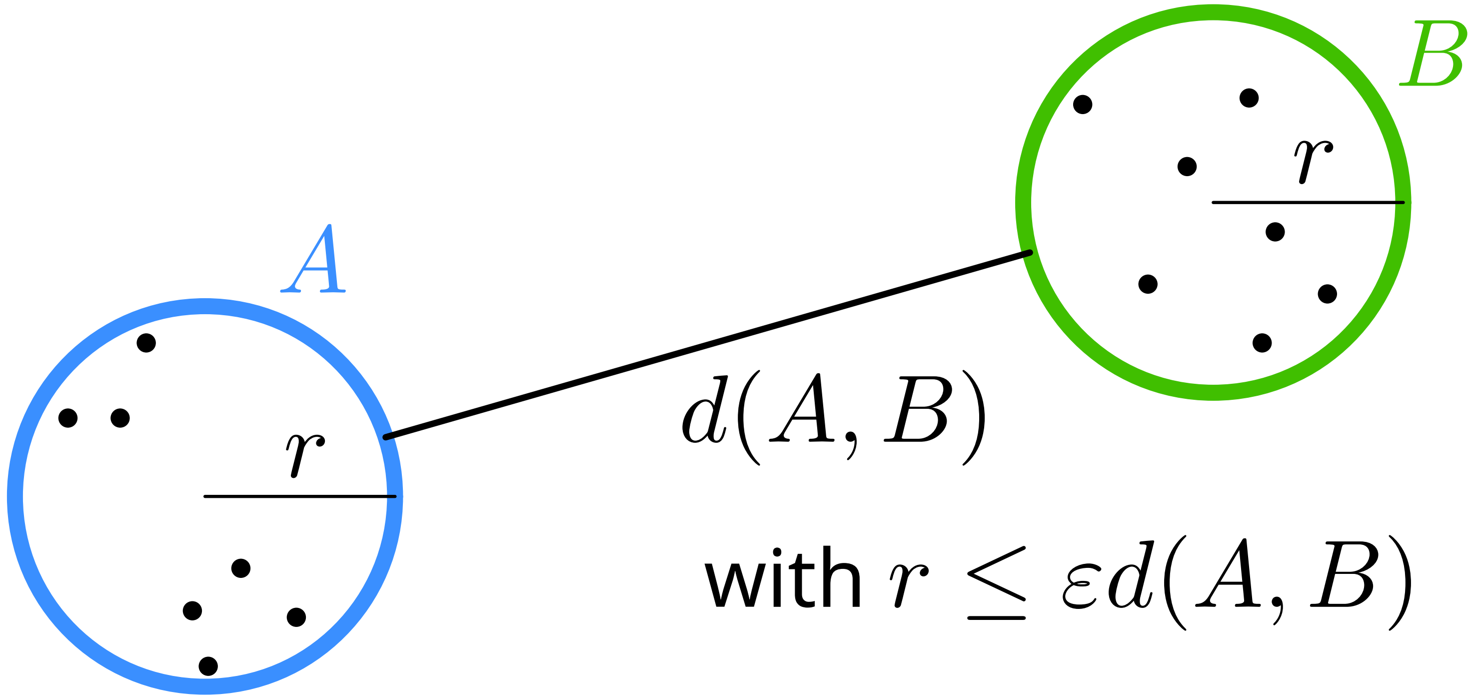
This requires $O(n)$-ANN $\rightarrow$ today

Motivation 2: The weight of the WSPD

Well-separated pair decomposition: Cover all pairs of points by $1/\varepsilon$-well-separated pairs of point sets $\{A, B\}$:



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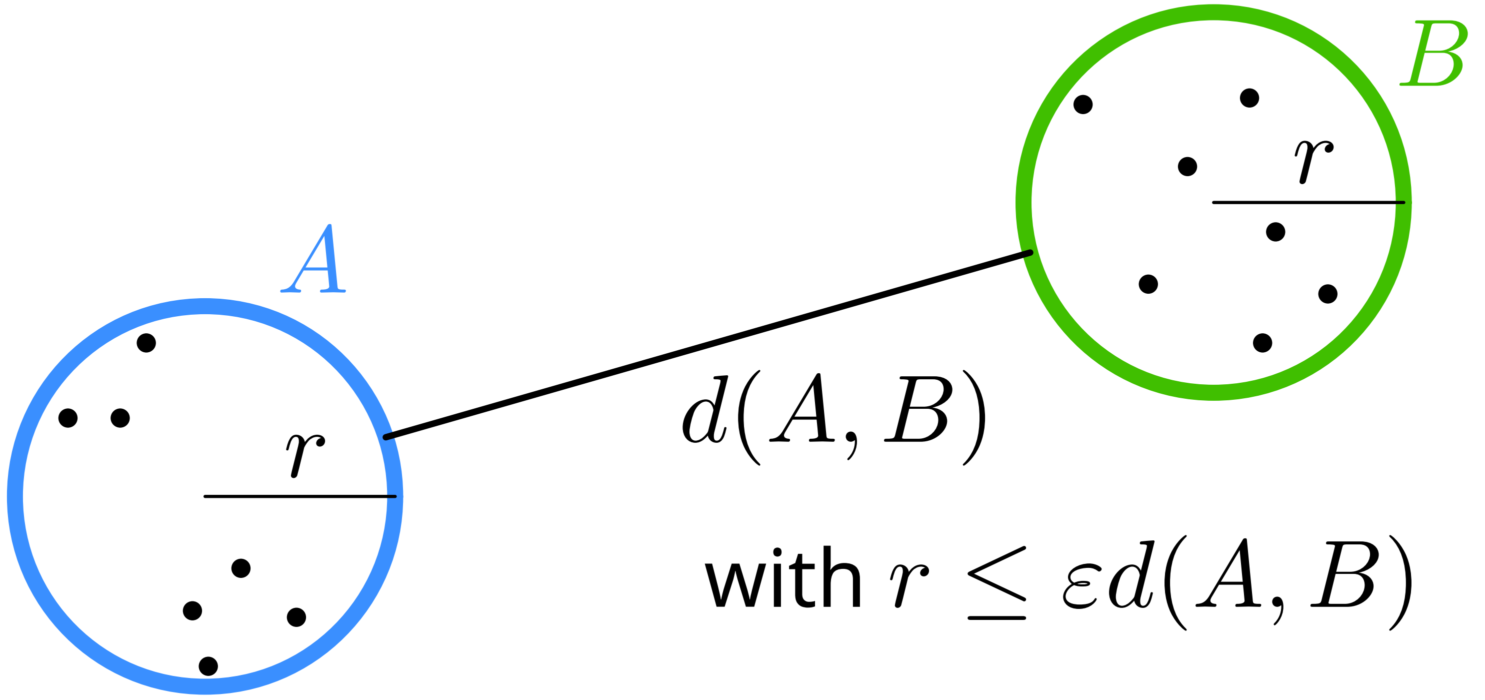

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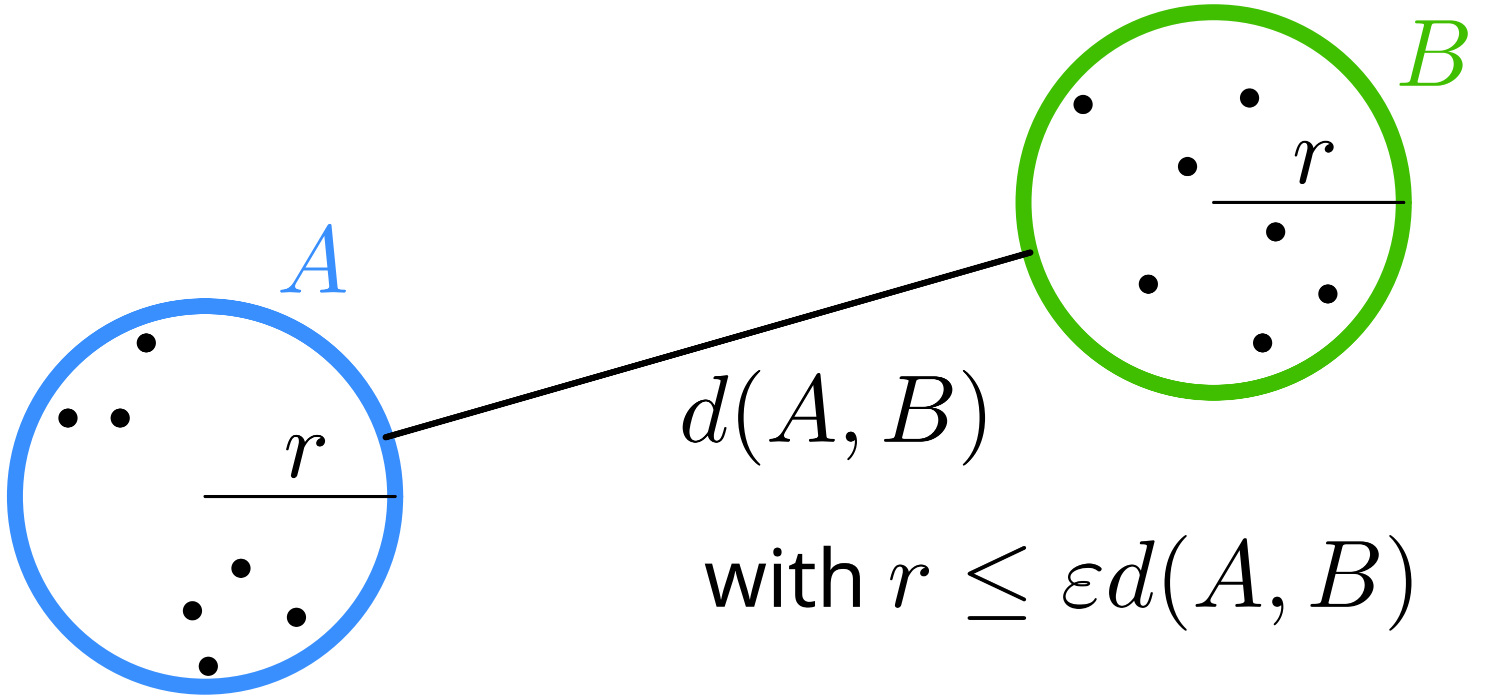

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but: might require $\sum_i |A_i| + |B_i| = \Theta(n^2)$ (weight of WSPD)

Overview

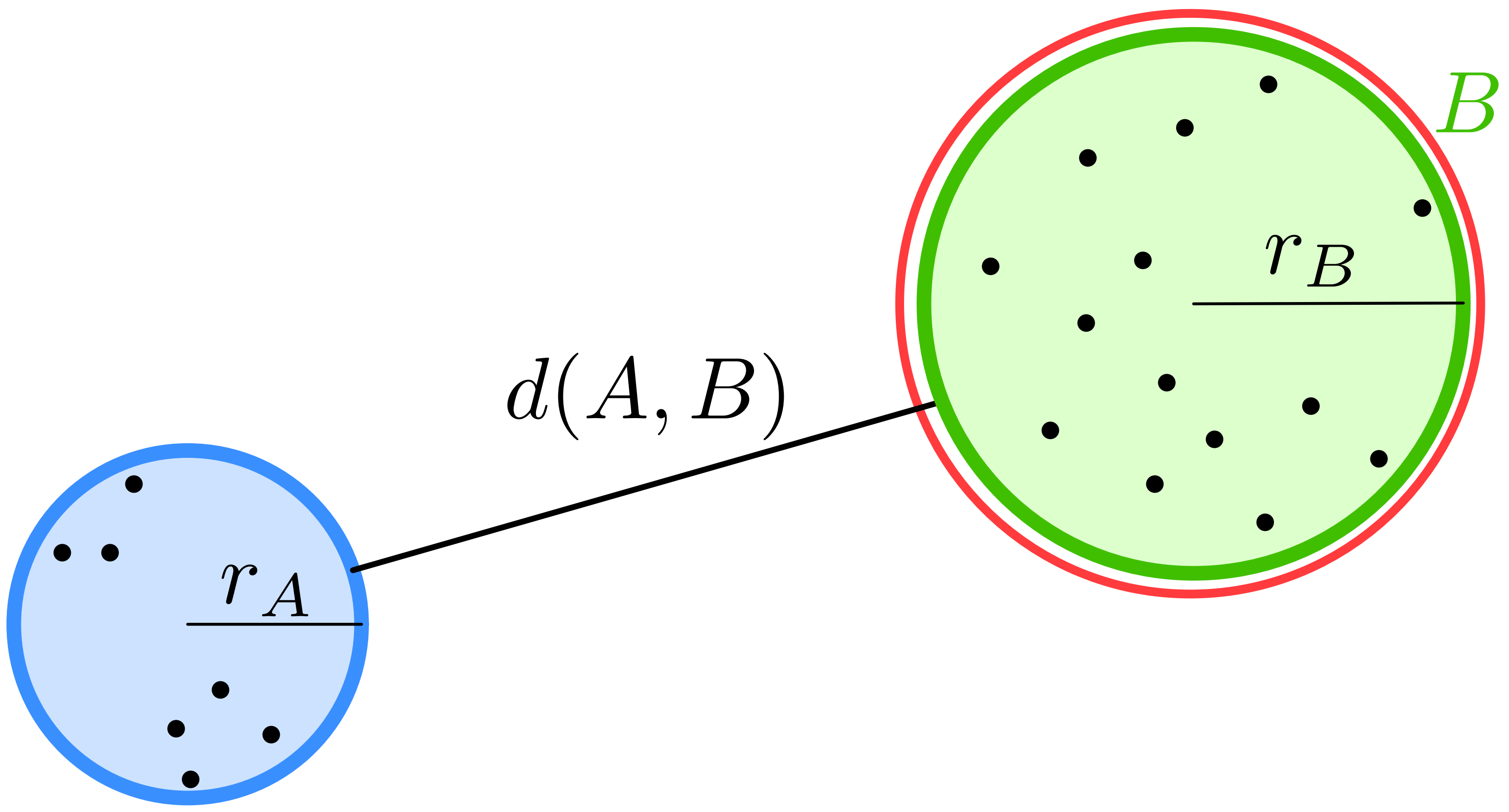
Semi-separated pair decomposition (SSPD)

Ring separator tree: n-semi-separated pair decomposition

Ring separator tree: $O(n)$-ANN

$(1/\varepsilon)$-semi-separated pair decomposition

Semi-Separated Pairs



$1/\varepsilon$-well-separated pair: $\max(r_A, r_B) \leq \varepsilon d(A, B)$

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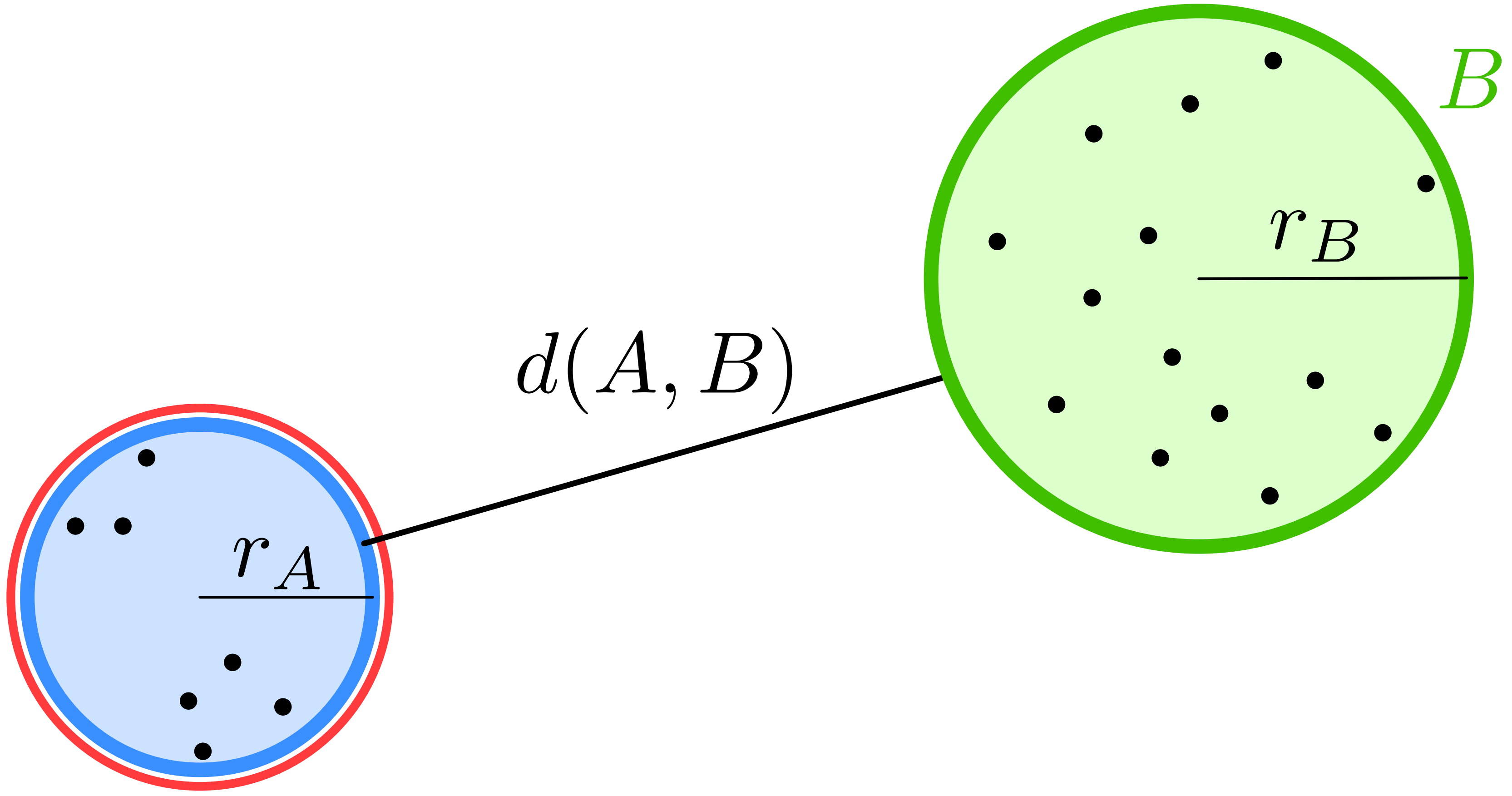

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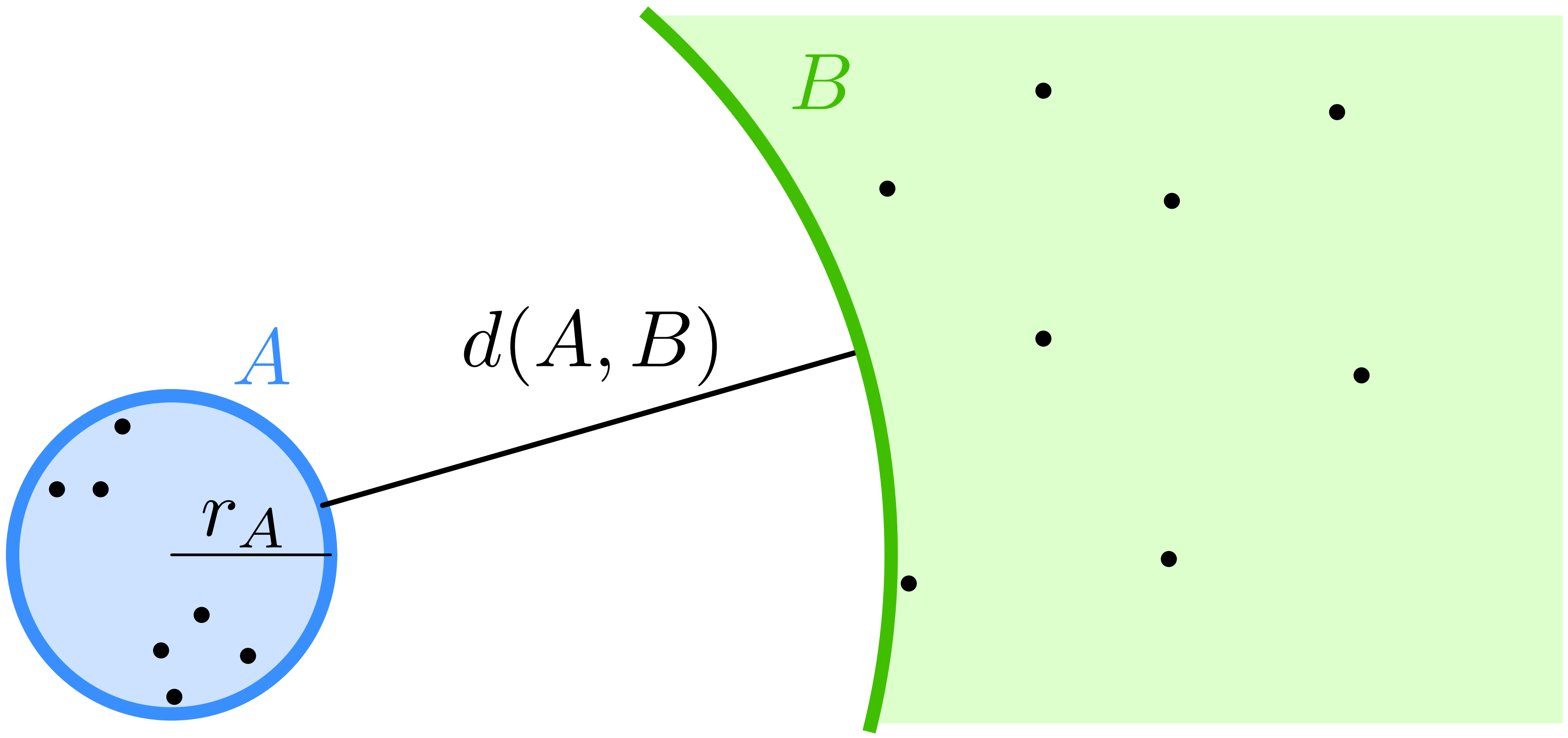

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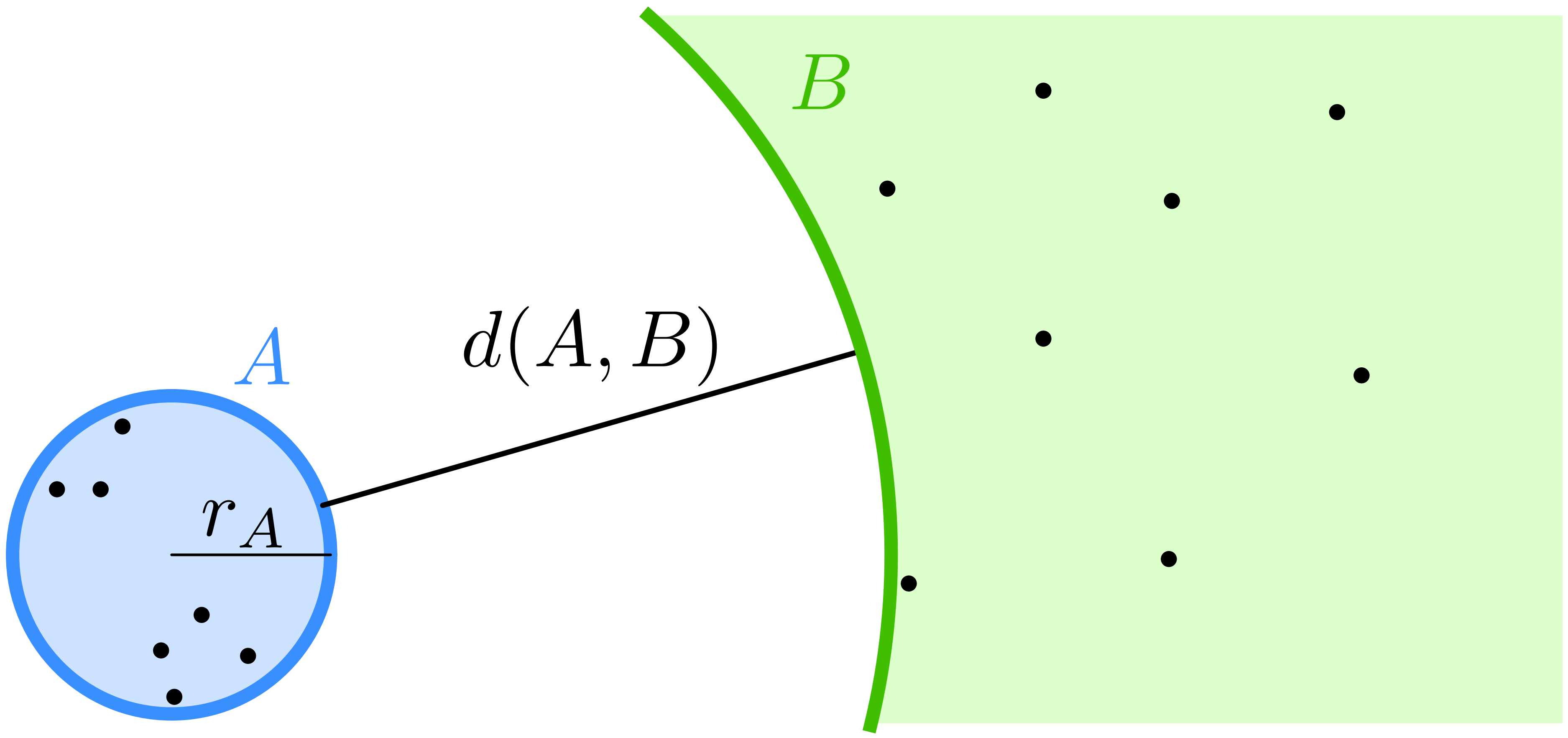


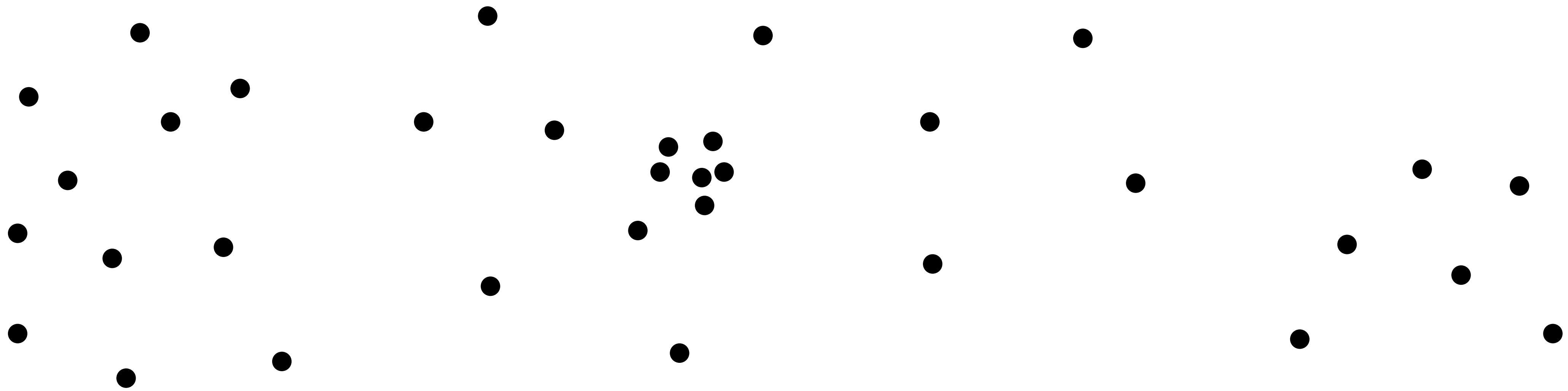
ring separator:

- ball $b = b(p, r)$ containing $\geq n/c_1$ points
- no point in $b(p, r(1 + 1/n)) \setminus b$
- $\geq n/c_2$ points outside $b(p, 2r)$

Computing a ring separator

1. Compute $b = b(p, \alpha)$: 2-approximation of smallest ball containing n/c_1 points

(c_1 to be determined later)



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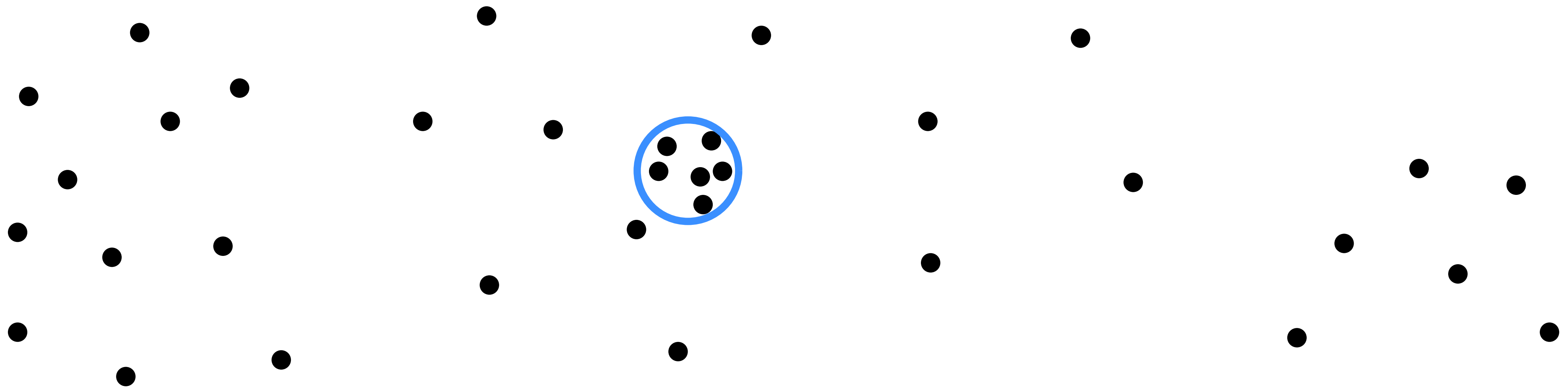

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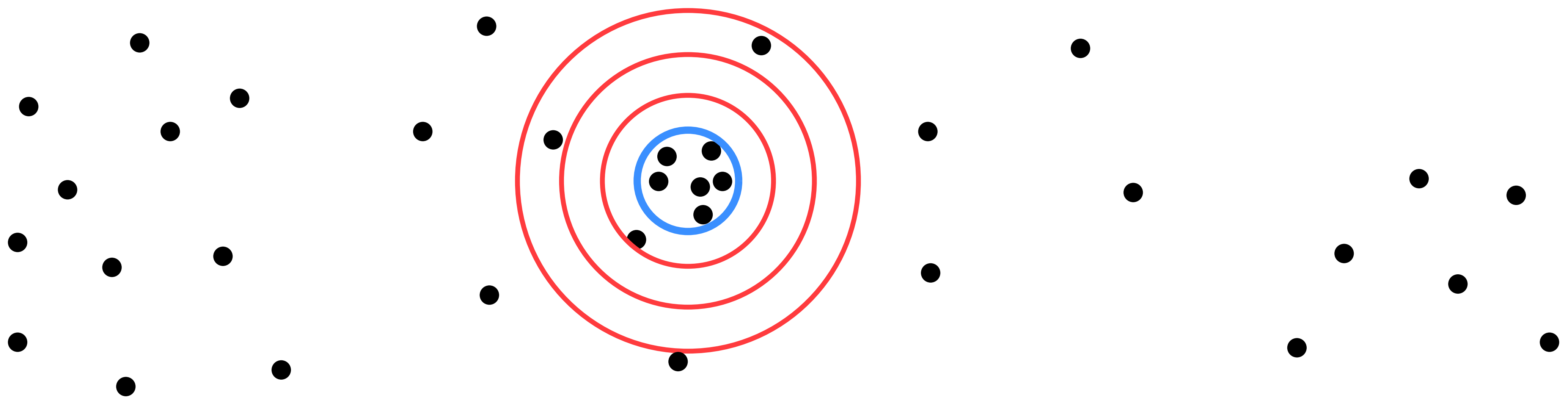

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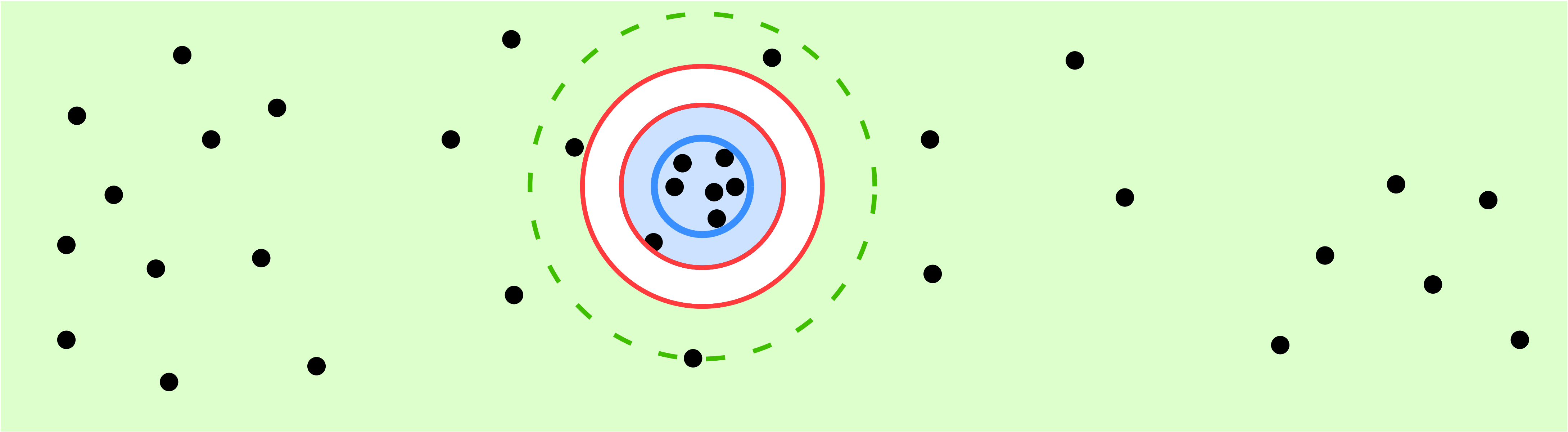
3. Find empty ring r_i and return b_{i-1}



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A $\Theta(n)$

B $\Theta(n \log n)$

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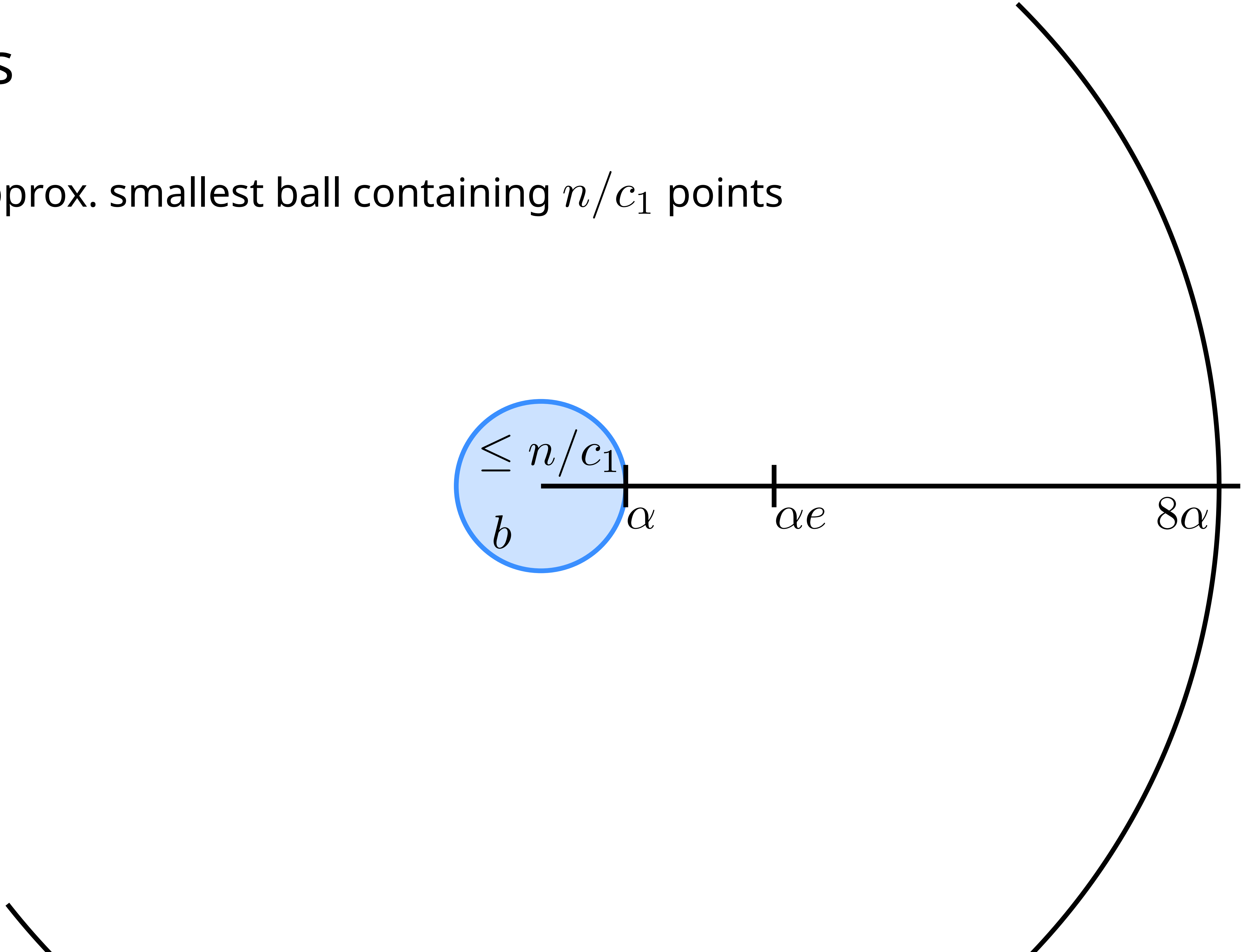

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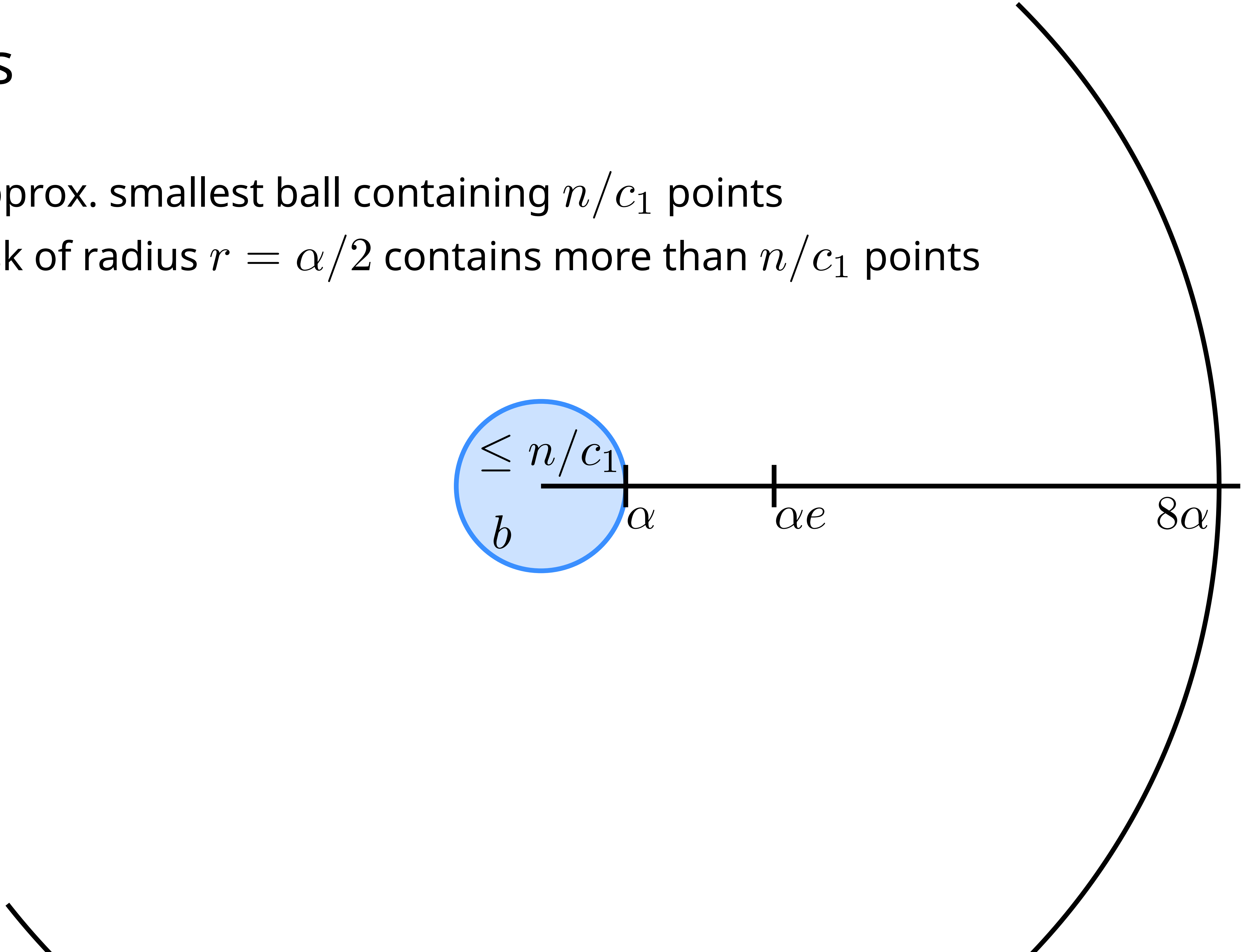

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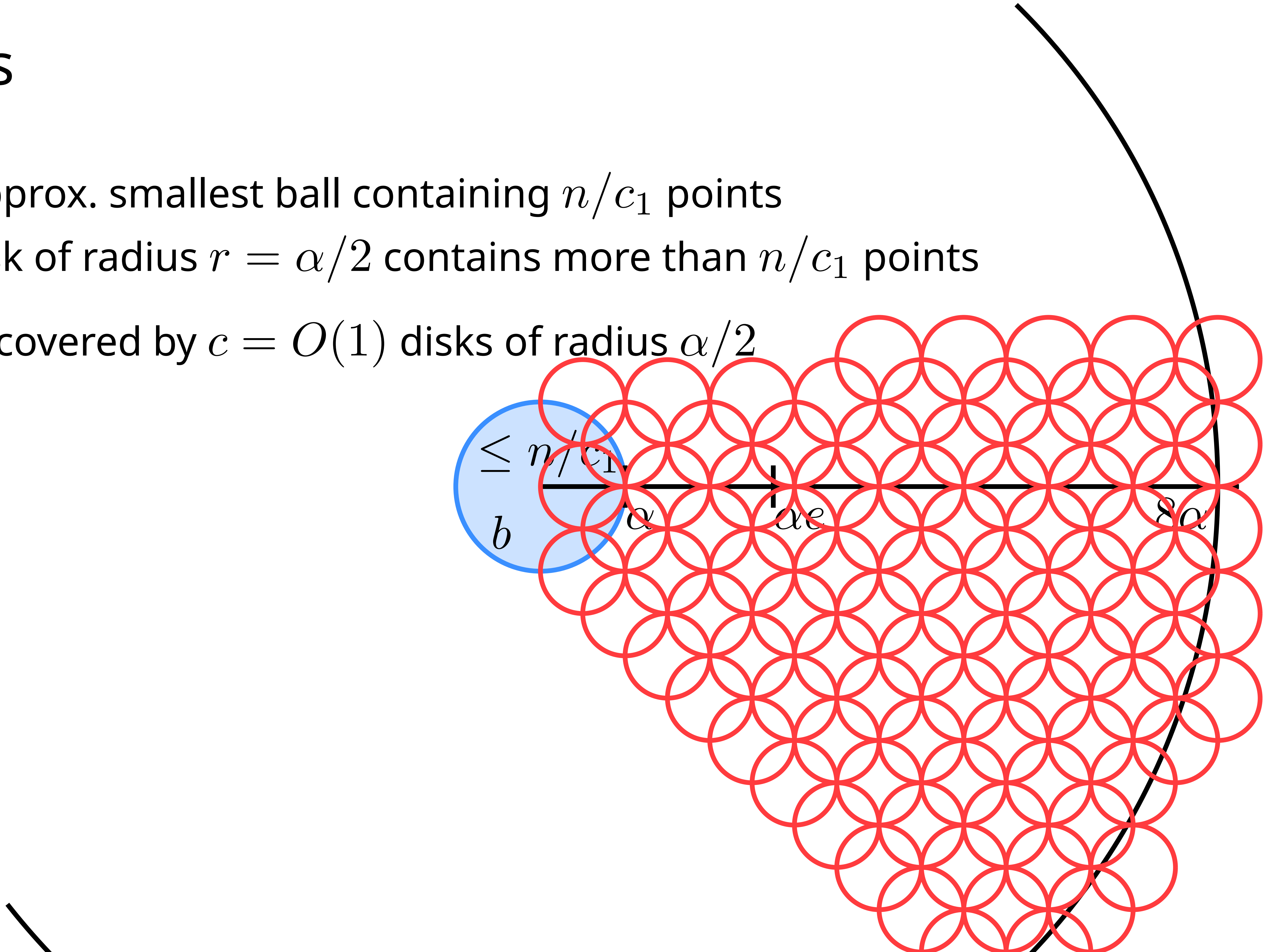

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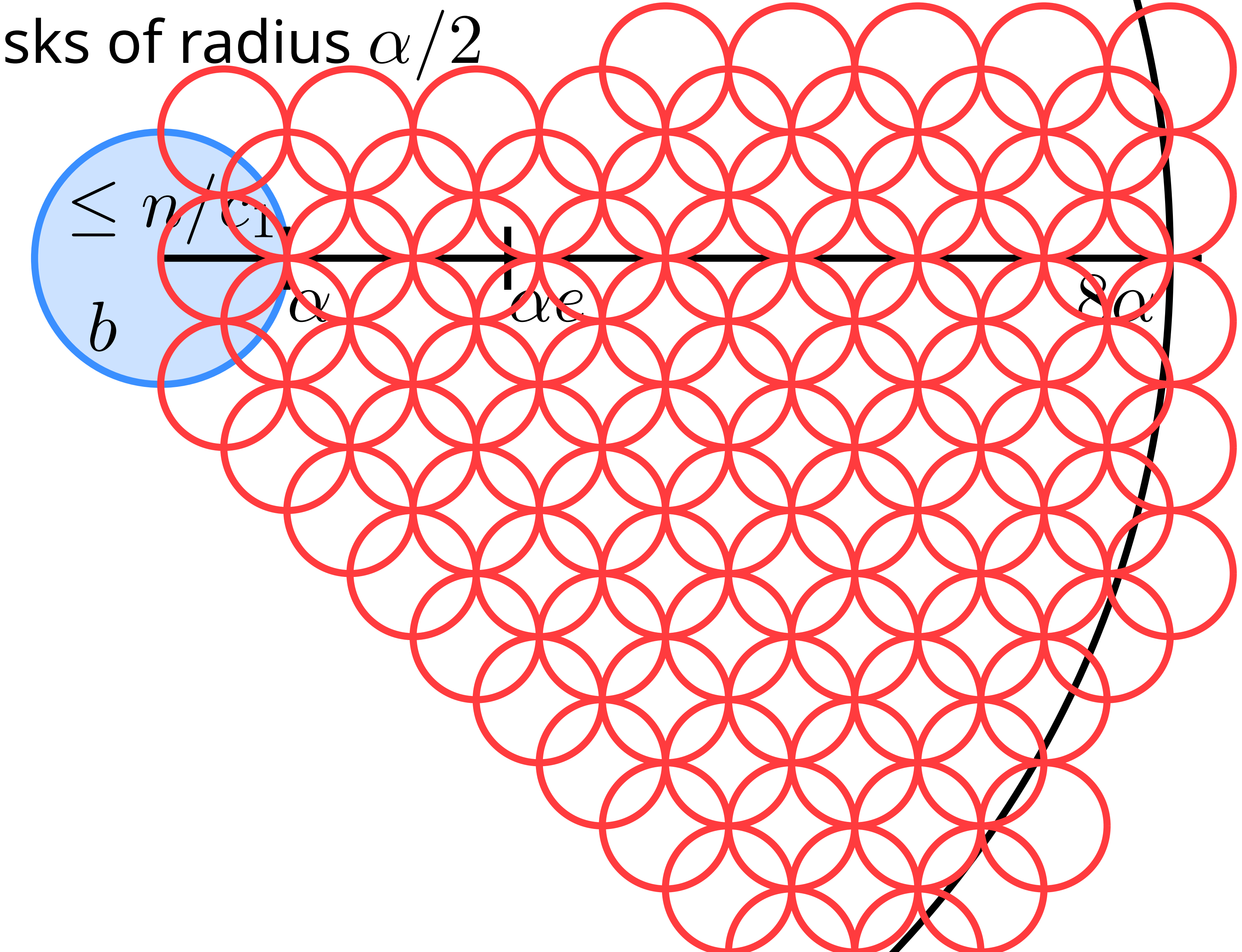

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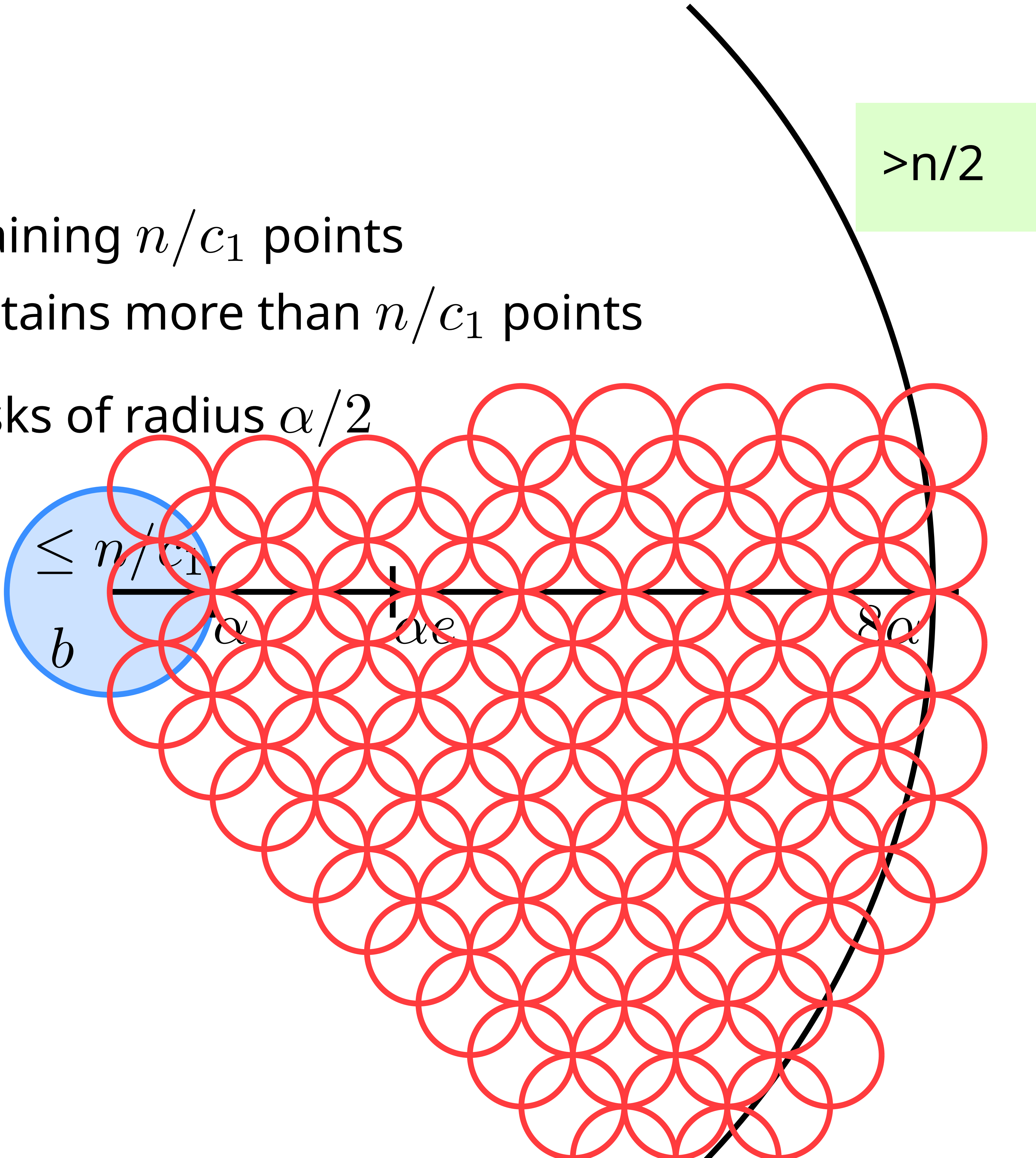

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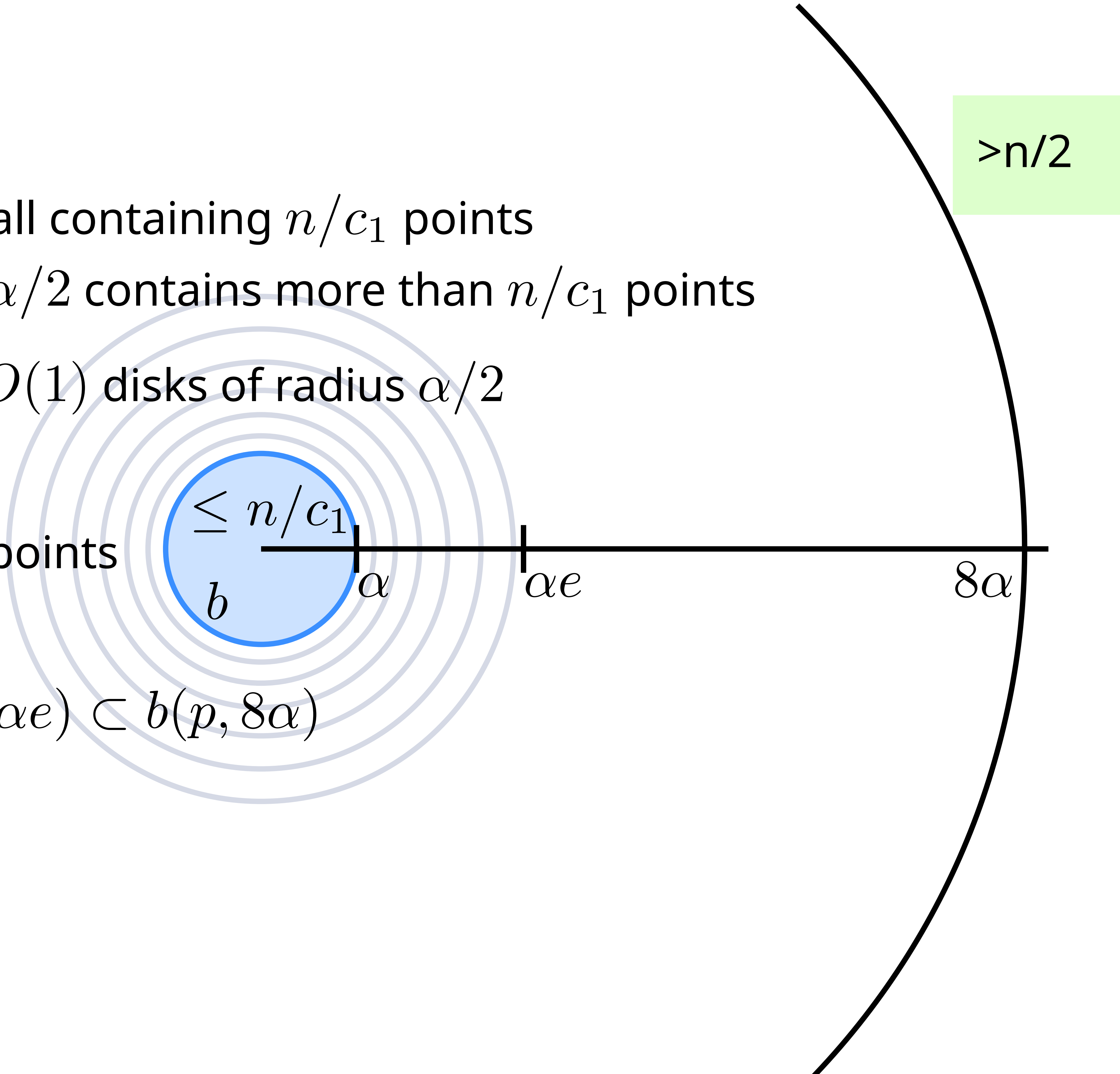

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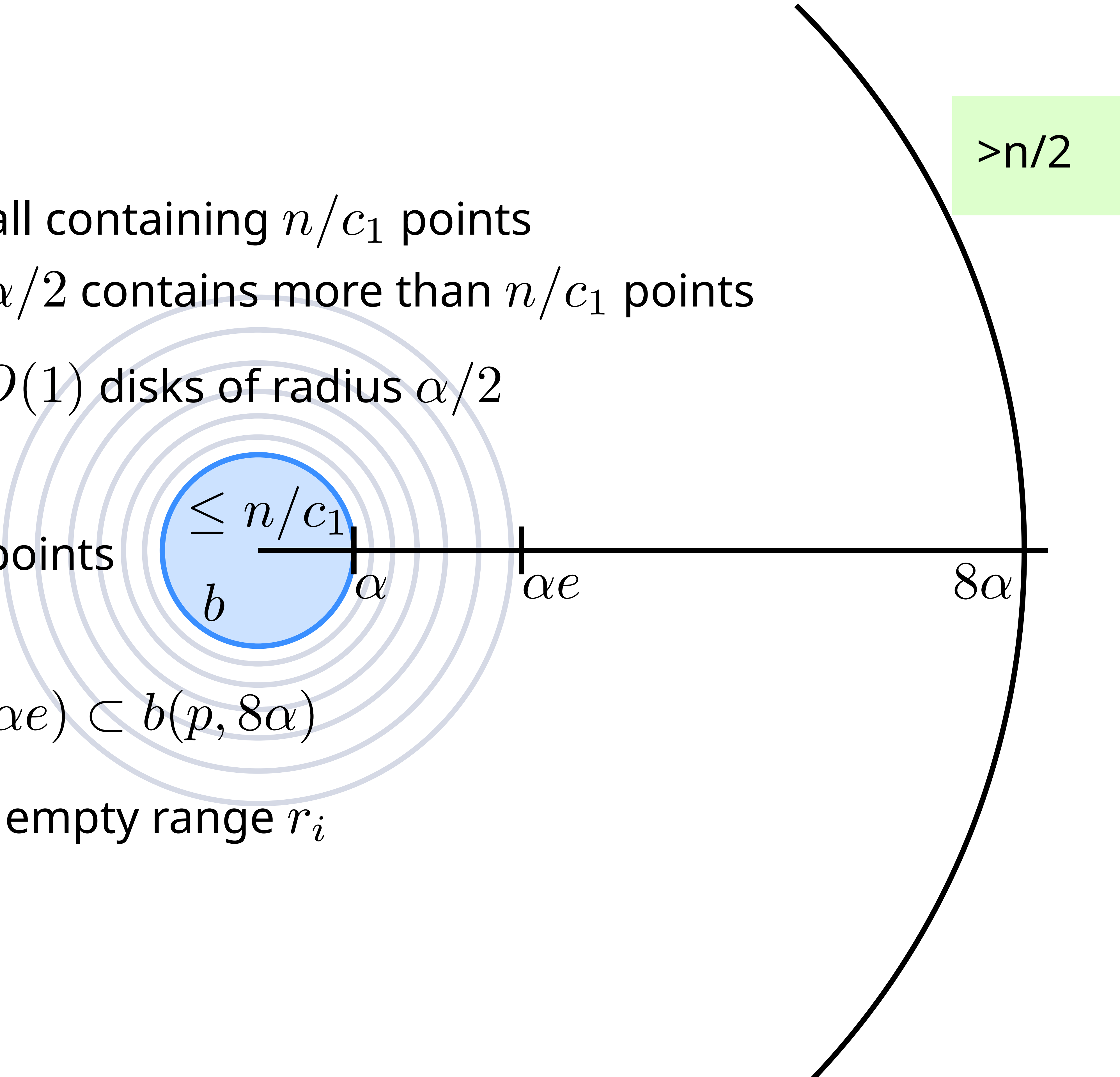

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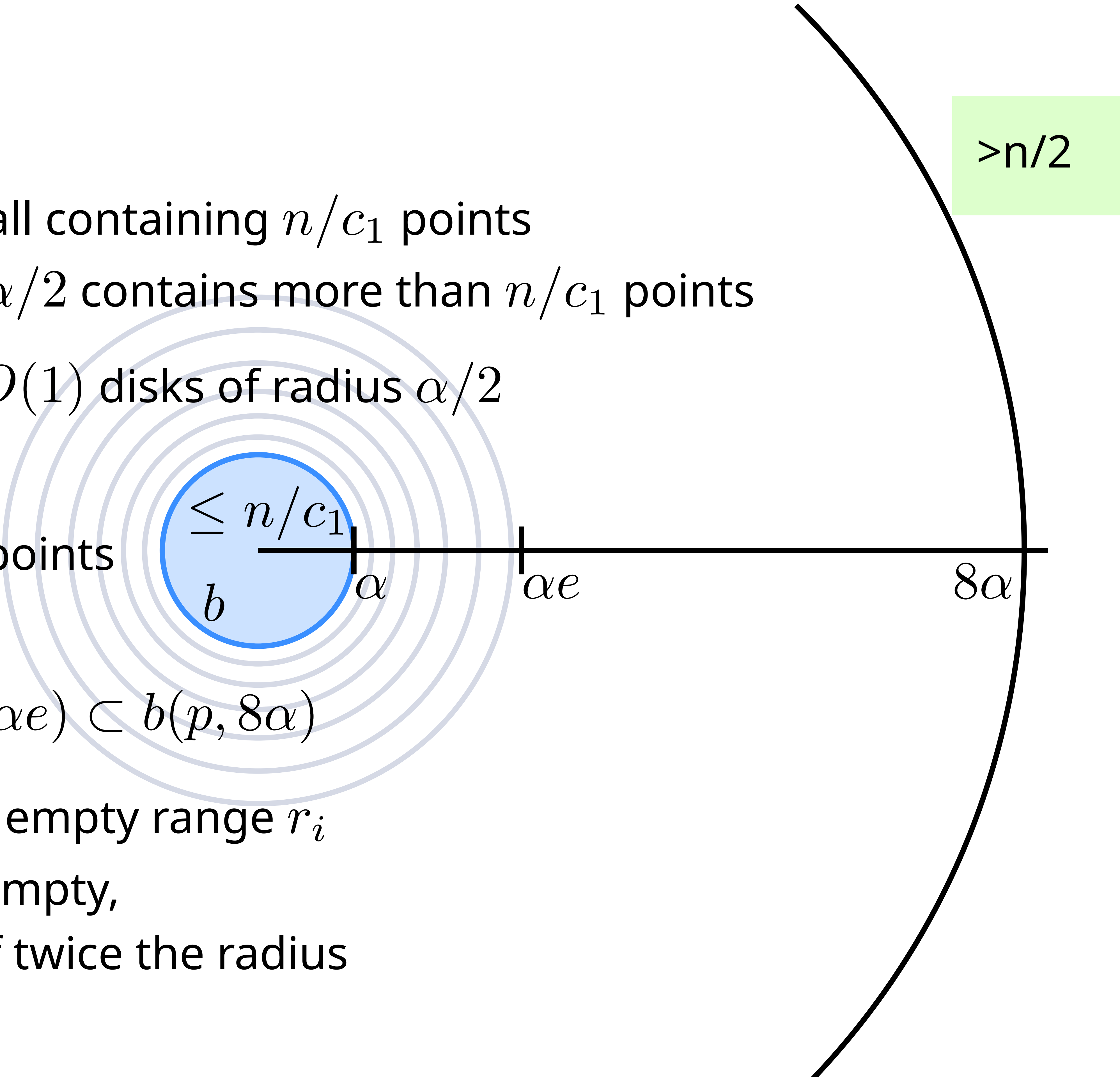

pigeonhole principle: There is an empty range r_i

b_{i-1} contains n/c_1 points, r_i empty,

$> n/2$ points outside of ball of twice the radius

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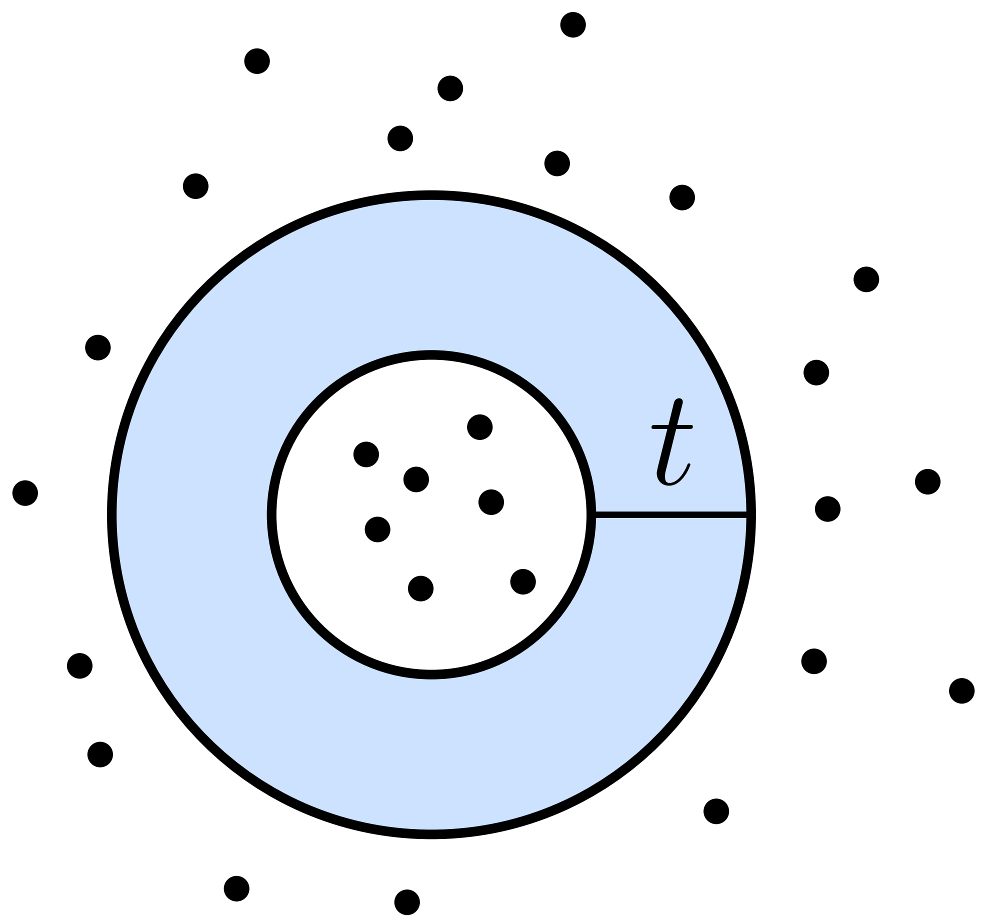

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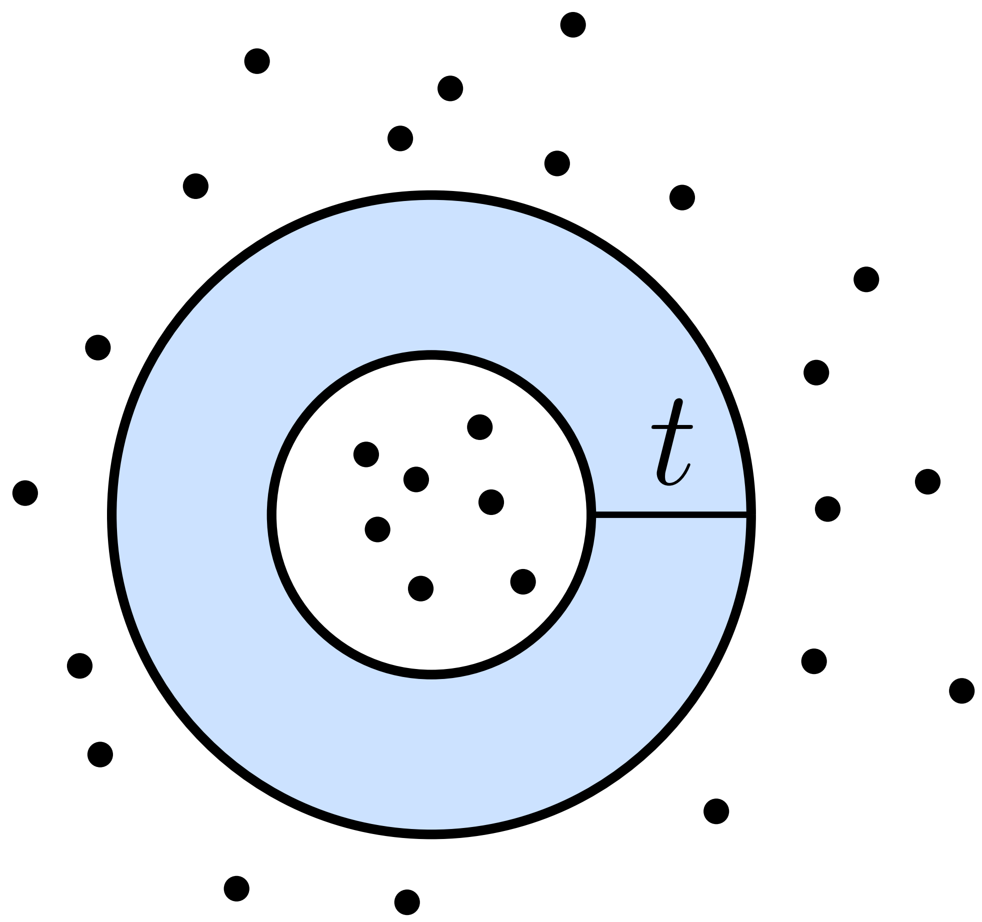

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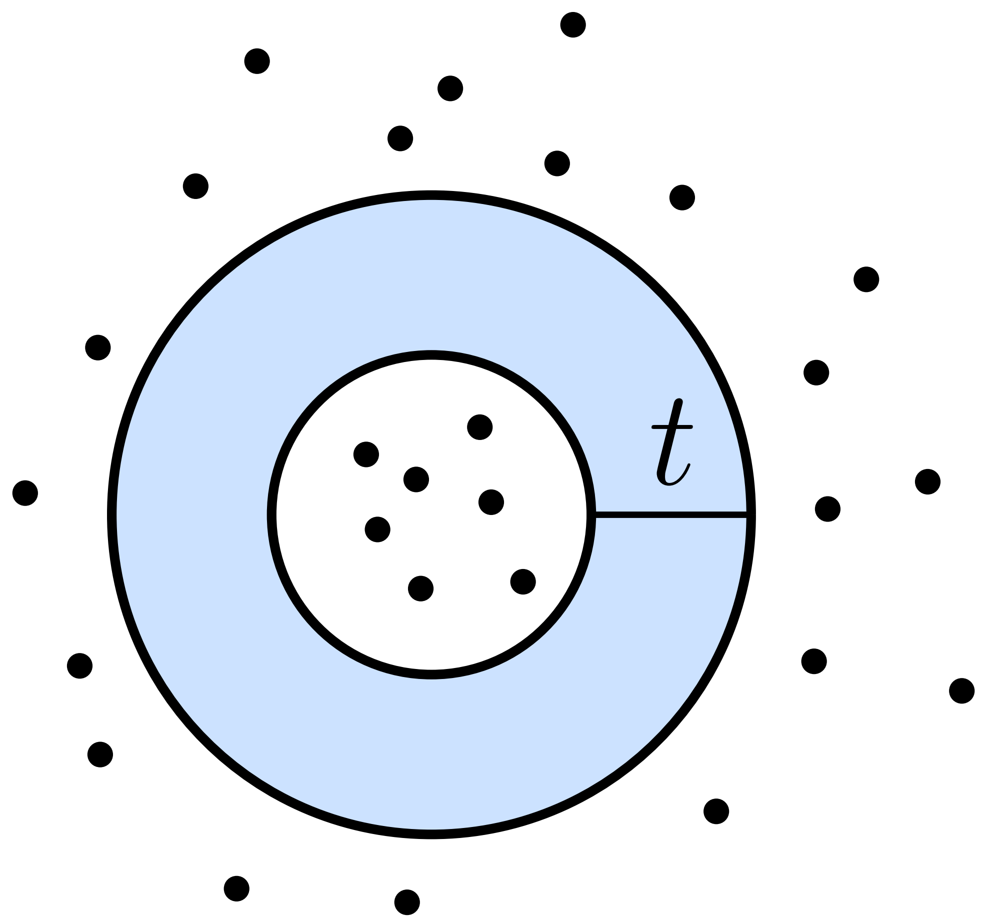

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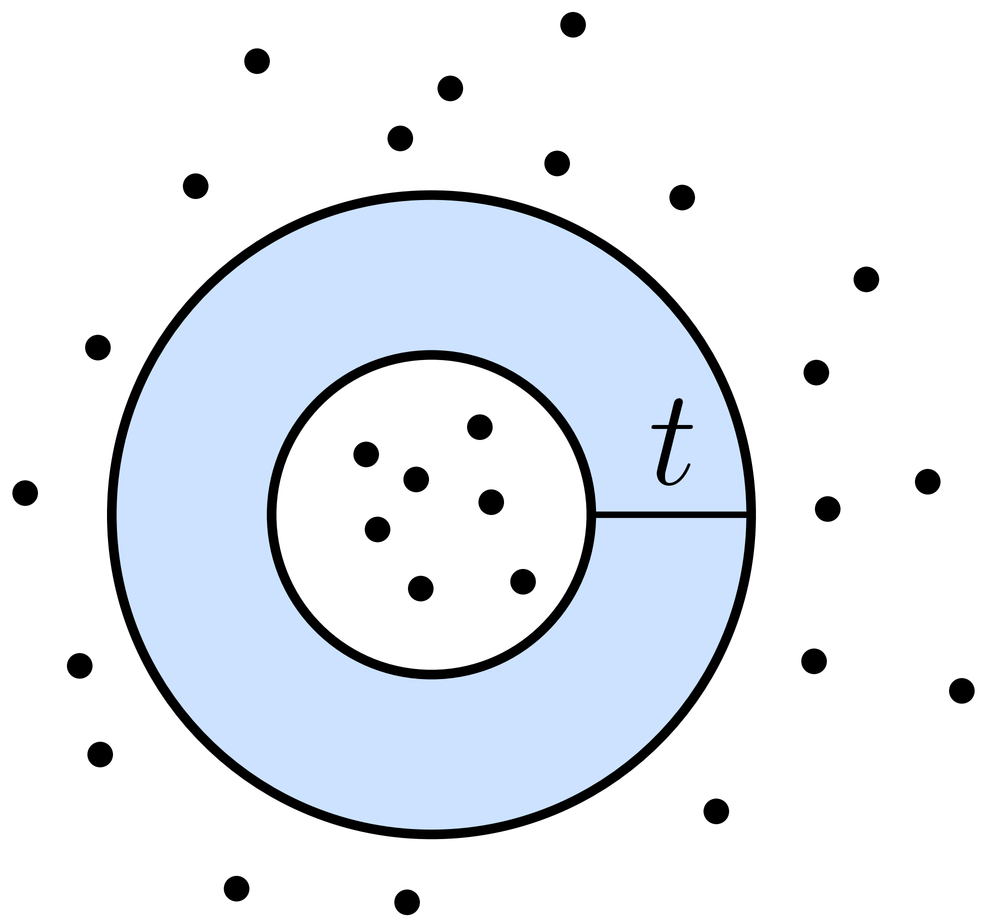
Apply algorithm for ring separator recursively



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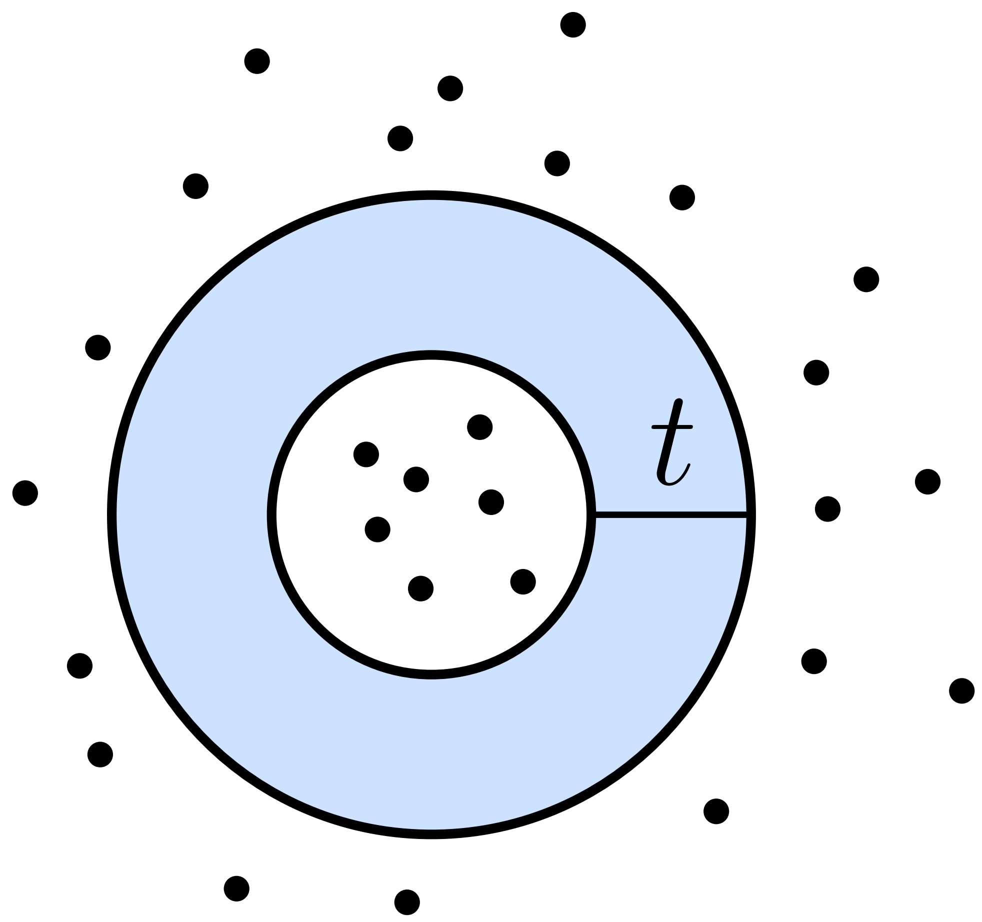
Result: A $\frac{1}{n}$-ring separator tree

A n-semi-separated pair decomposition of weight $\Theta(n \log n)$



Overview

Semi-separated pair decomposition (SSPD) ✓

Ring separator tree: n-semi-separated pair decomposition ✓

Ring separator: $O(n)$-ANN

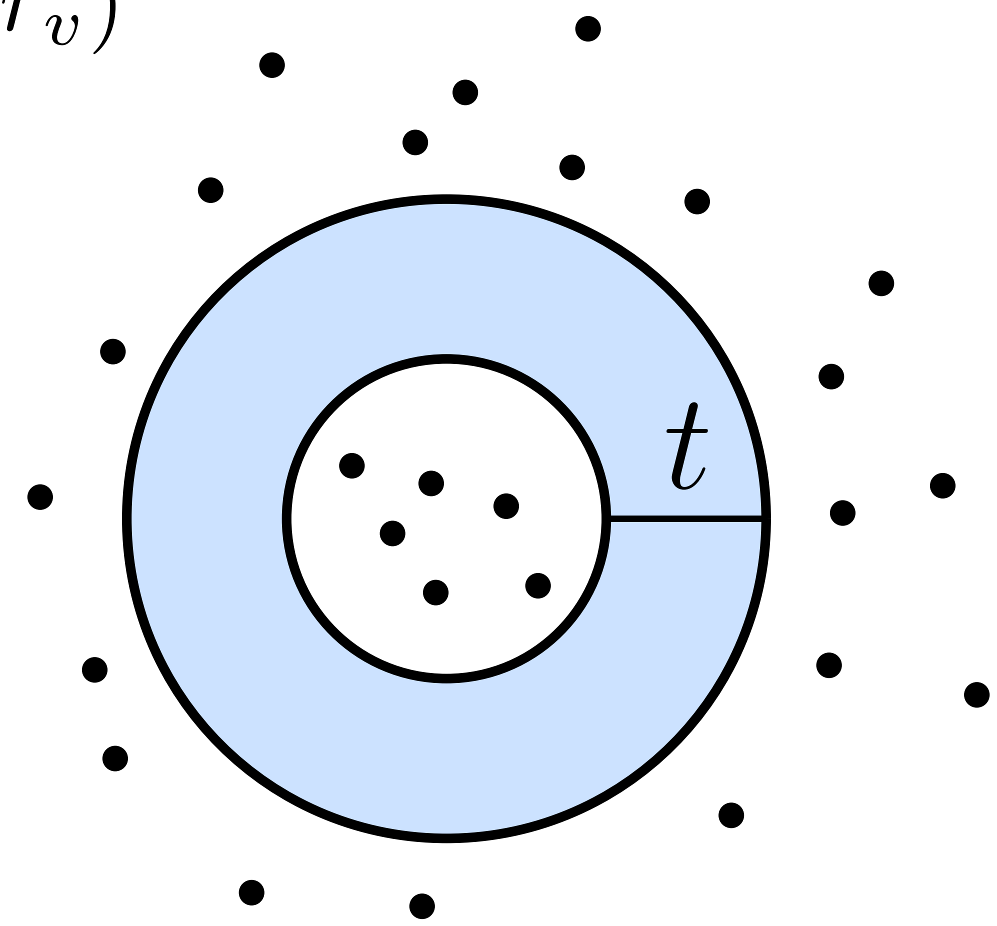
$(1/\varepsilon)$-semi-separated pair decomposition

Ring Separator Tree

For every node v we ensure:

There is a ball $b_v = b(c_v, r_v)$ s.t. all points of $P_v^{\text{in}} = P_v \cap b_v$ are in one child of v (the inner child)

All other points of P_v are outside $b(c_v, (1+t)r_v)$ and stored in the other child (the outer child)



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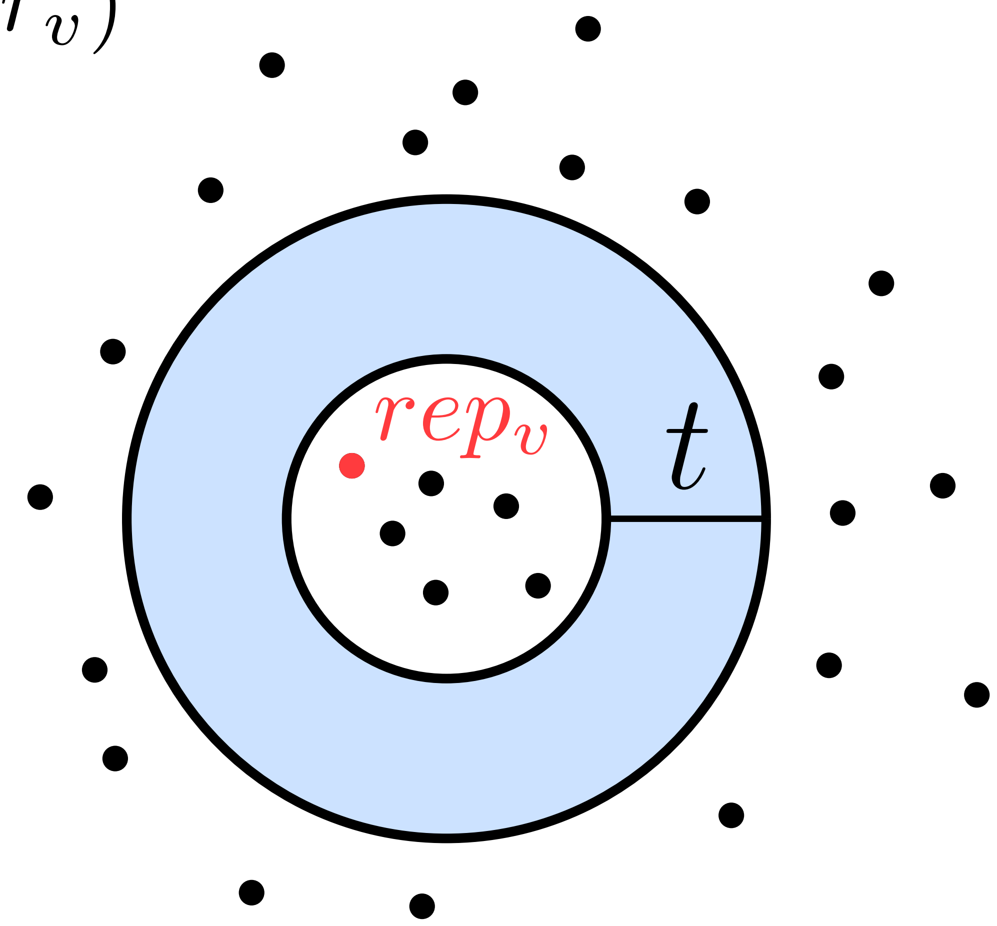
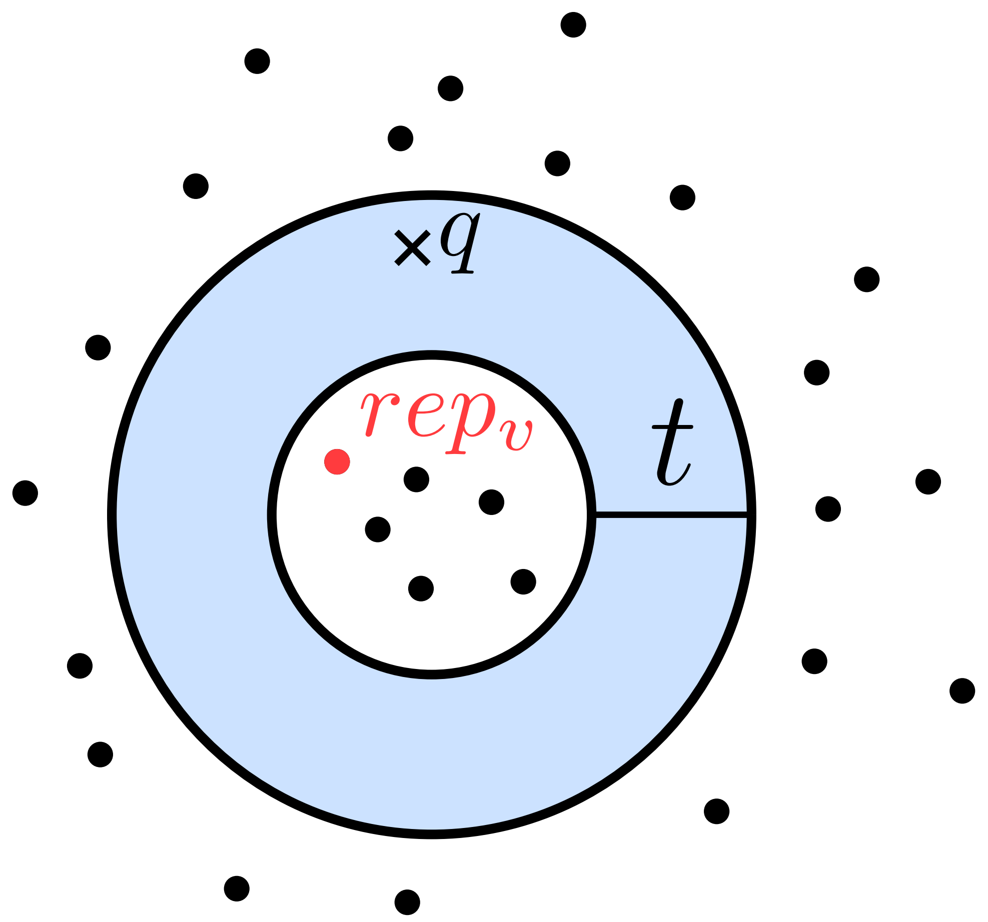
Store an arbitrary $rep_v \in P_v^{\text{in}}$ in v



ANN search procedure

Given query point q:



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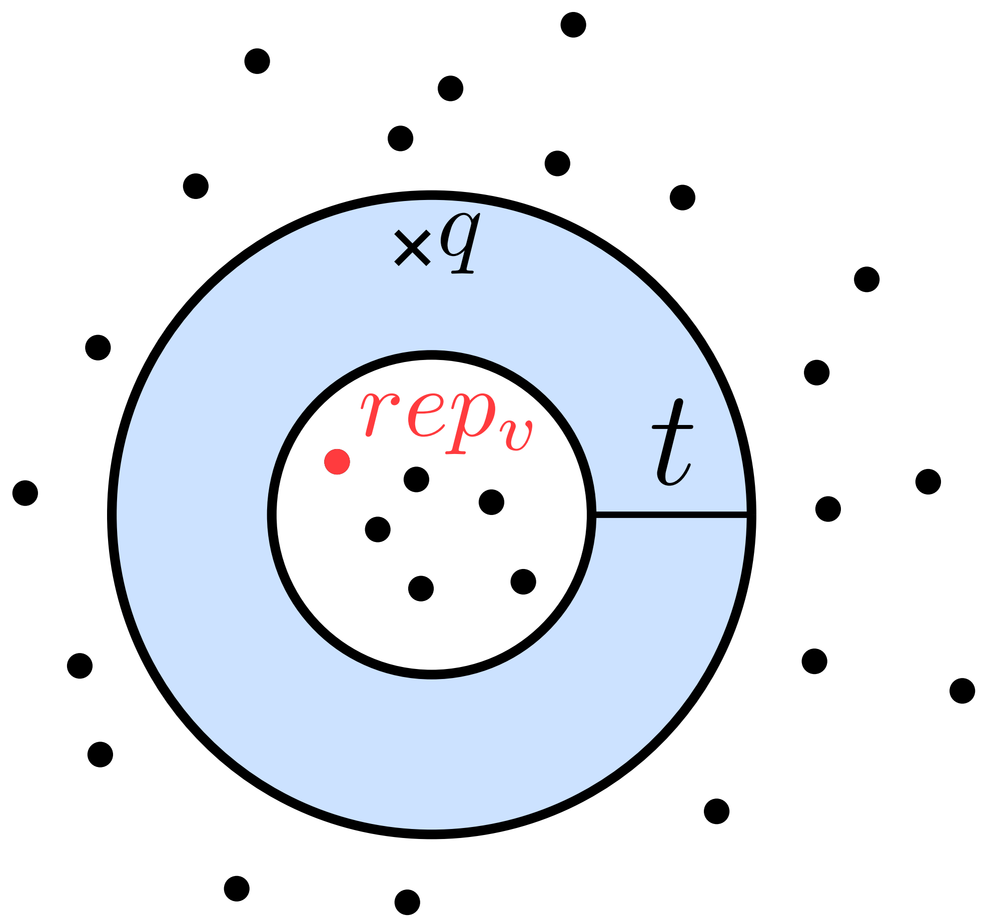

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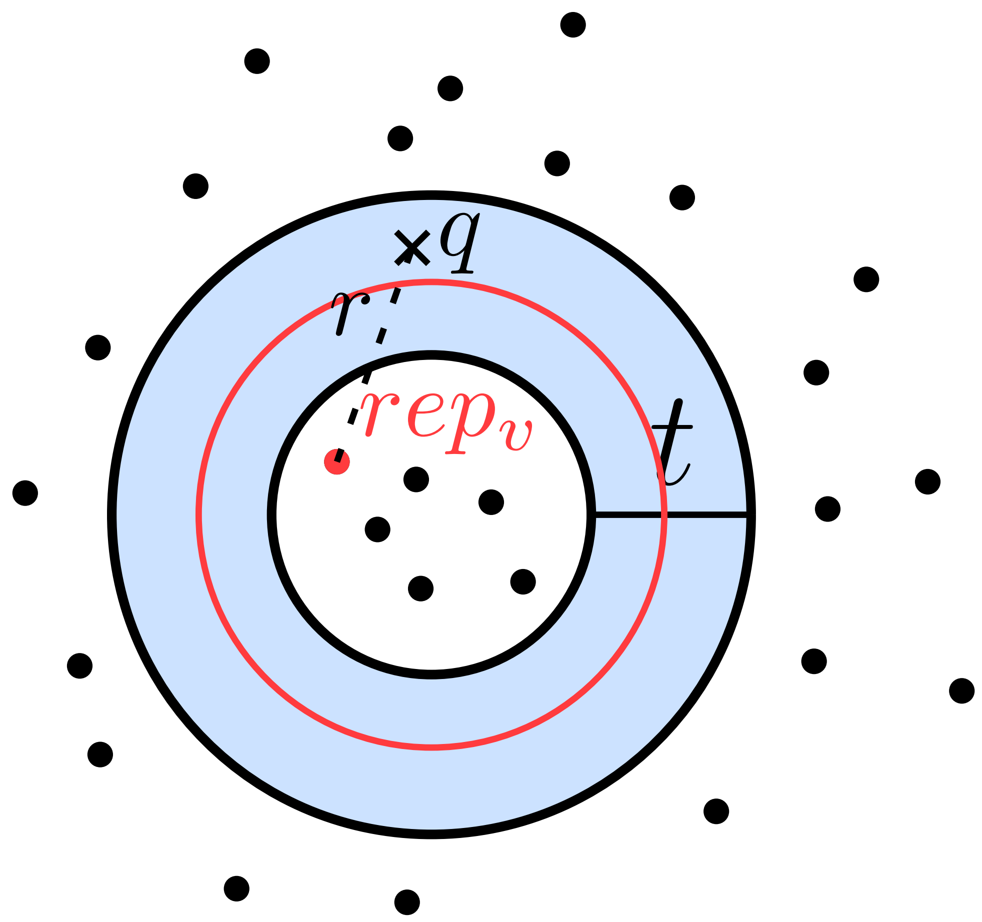

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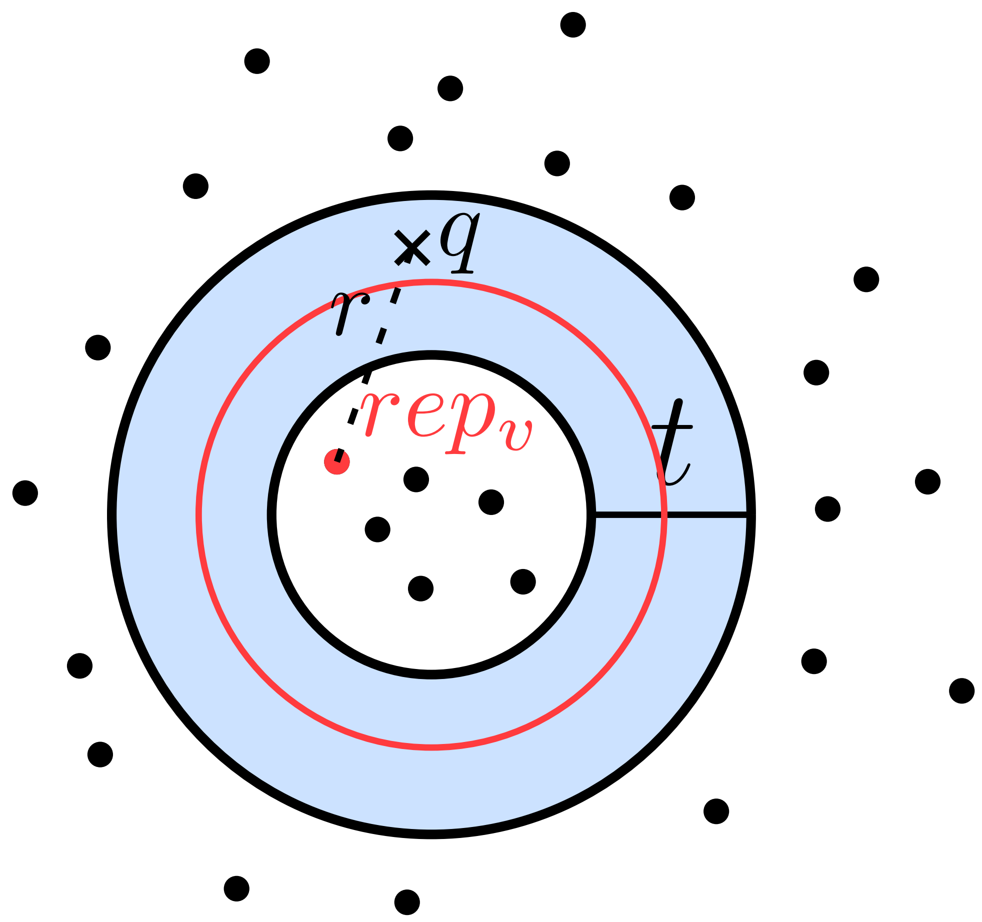

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if $\|q - c_v\| \leq r_{mid}$ then

$v =$ inner child of v

else

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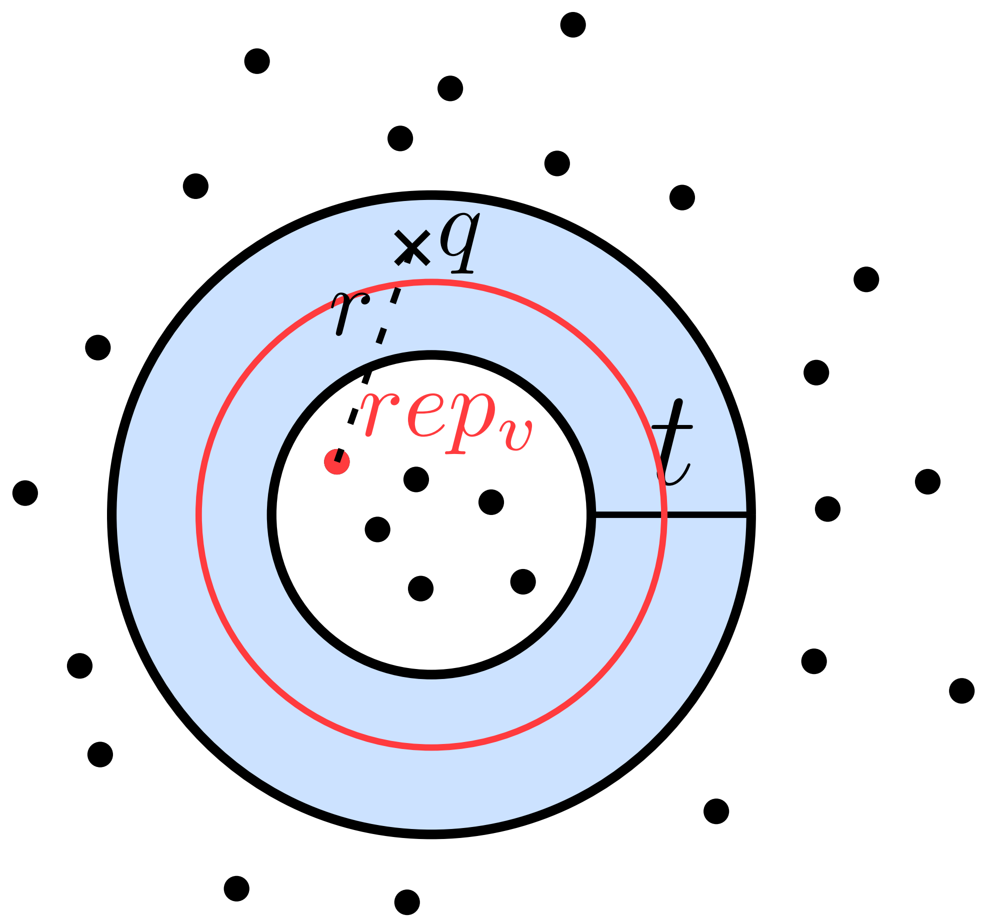
return r



ANN search procedure

Given query point q:

$v =$ root of T, $r = \infty$

while v is not a leaf:

$r = \min(r, \|q - rep_v\|)$

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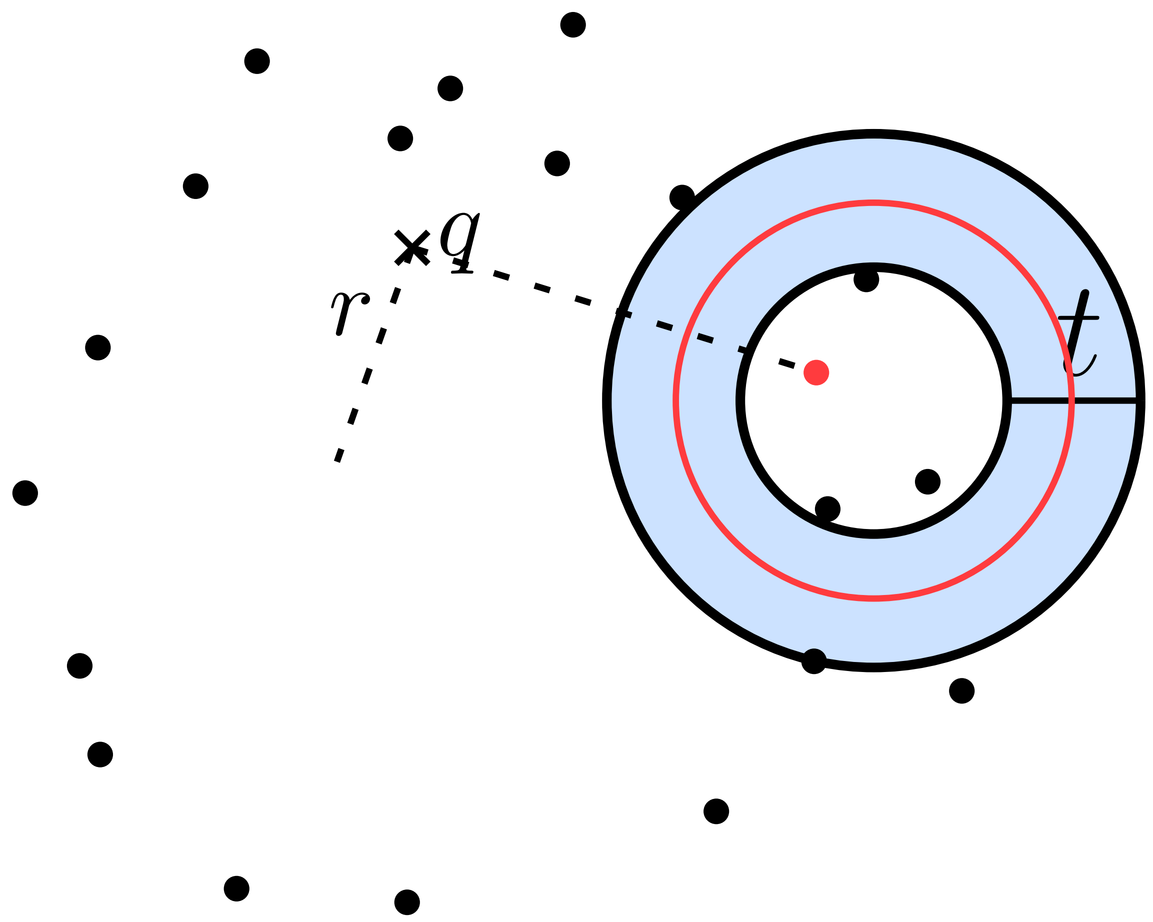

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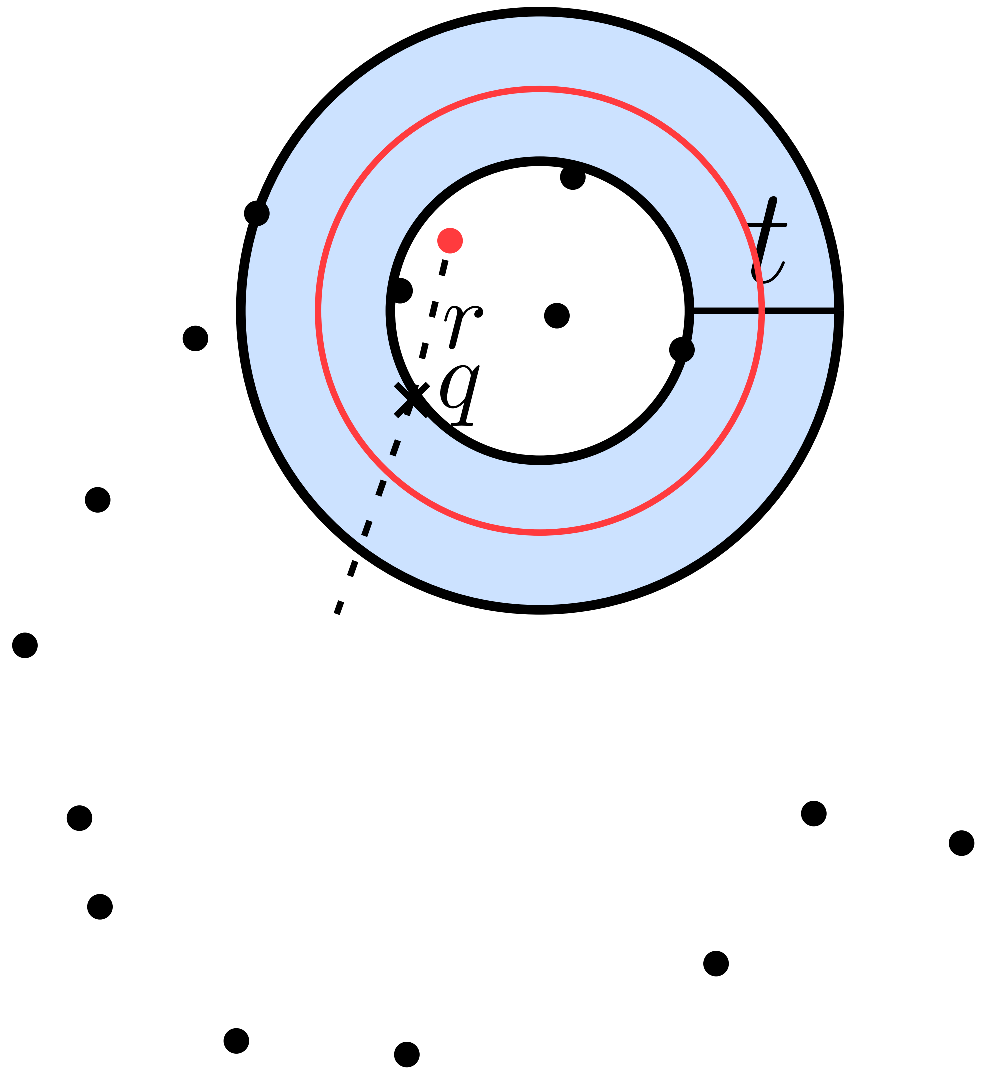

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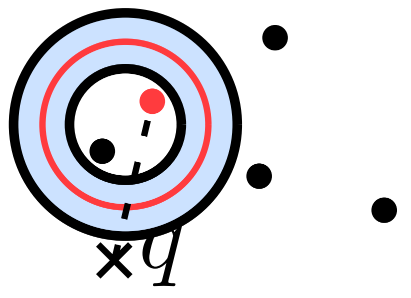

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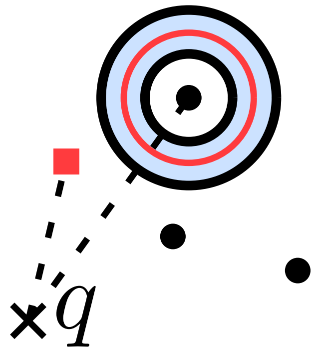

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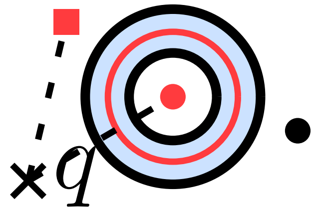

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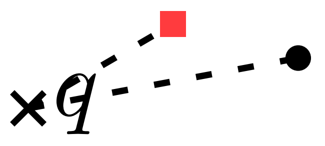

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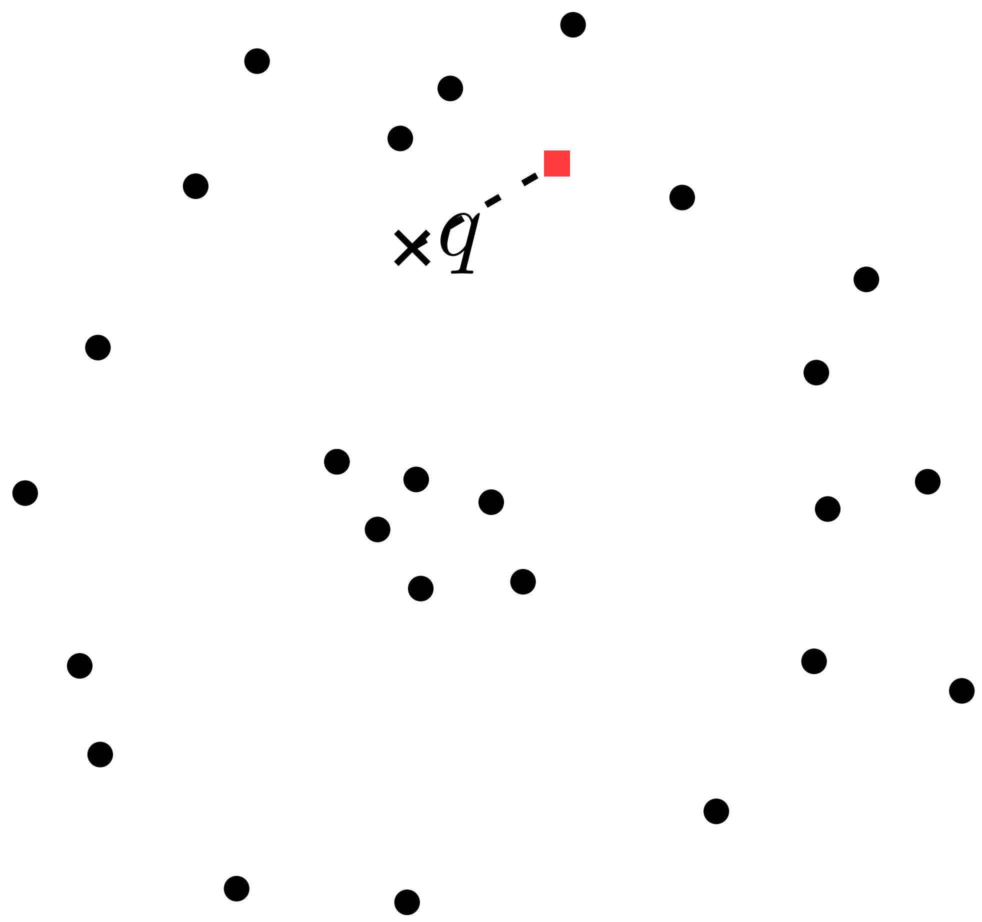
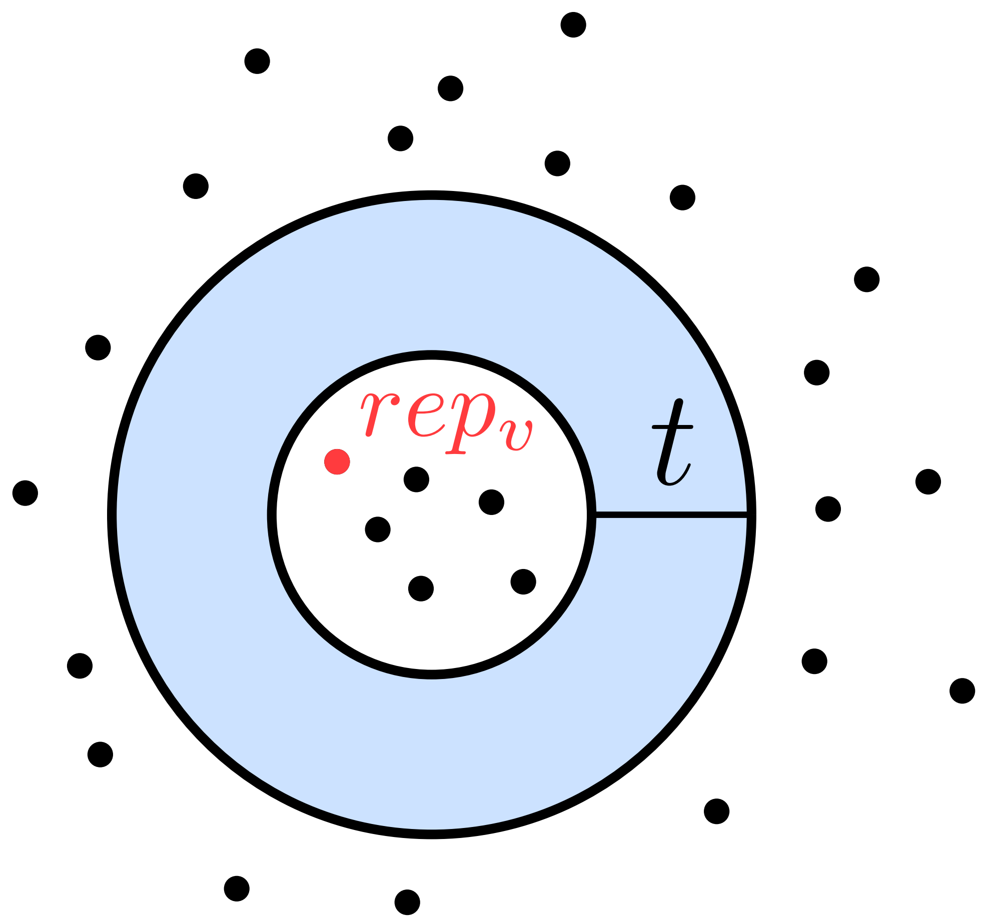


Intuition of correctness

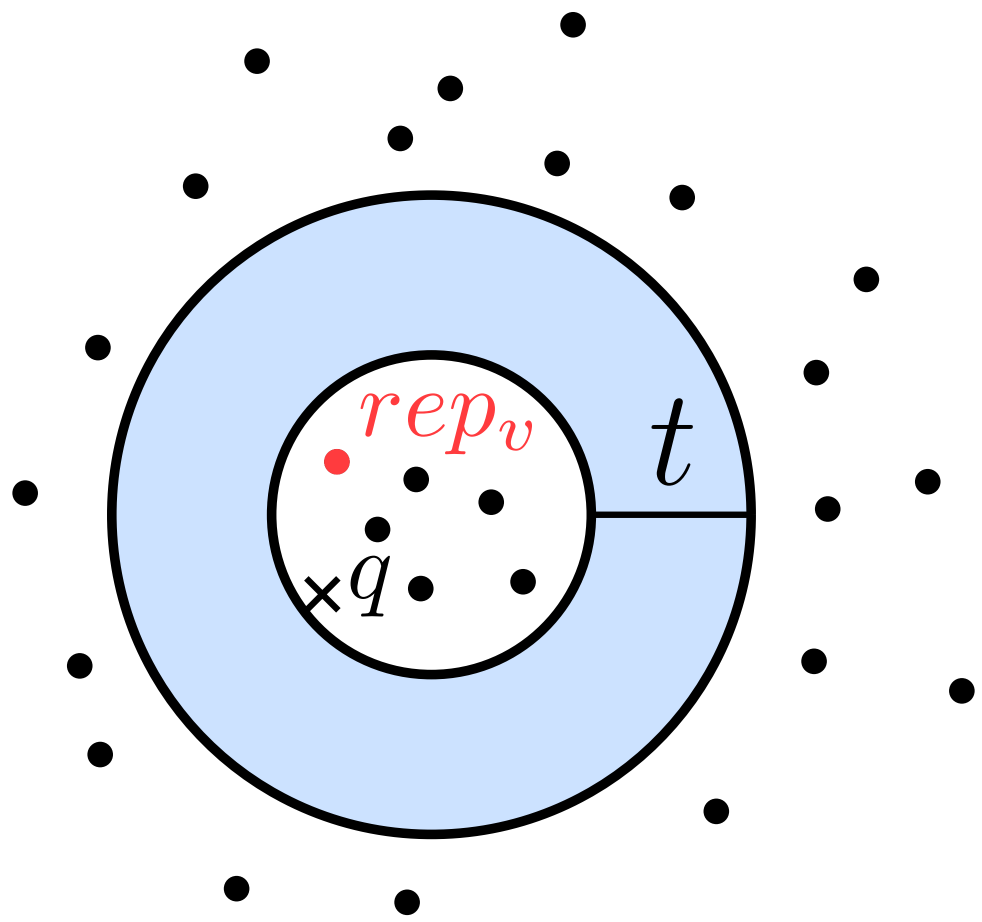
Case distinction:



Intuition of correctness

Case distinction:

1.) q in b_v



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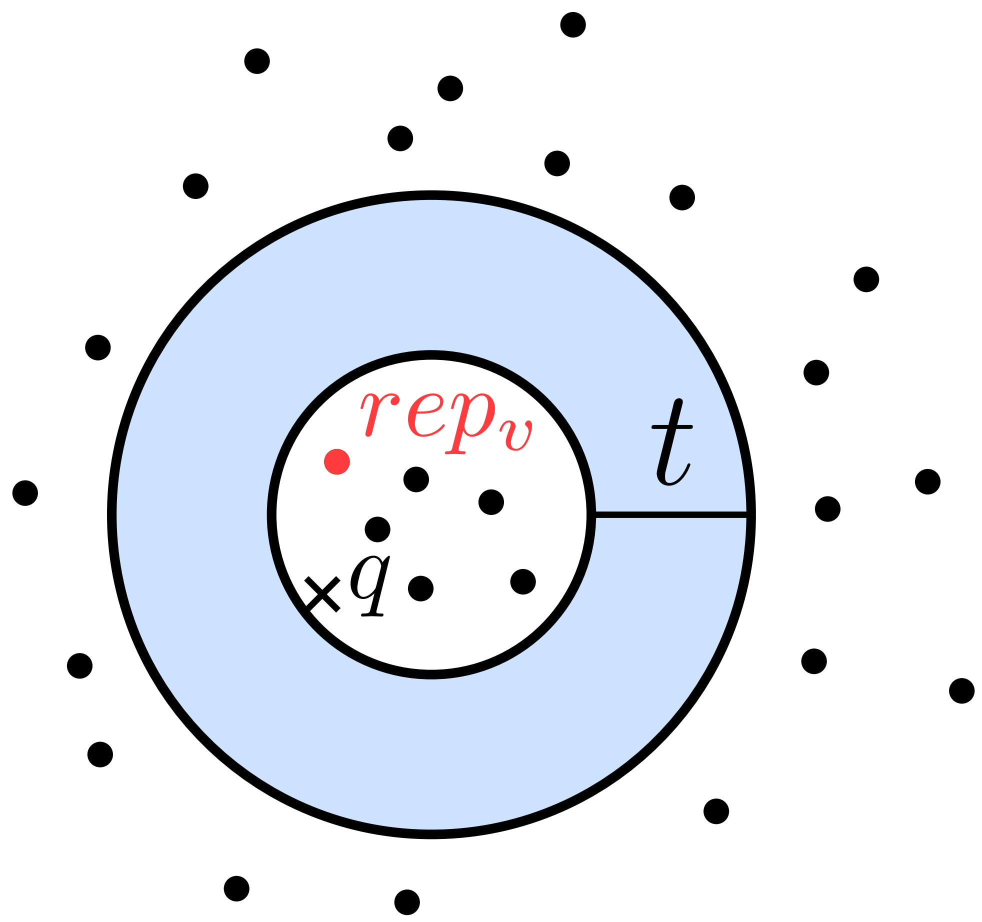

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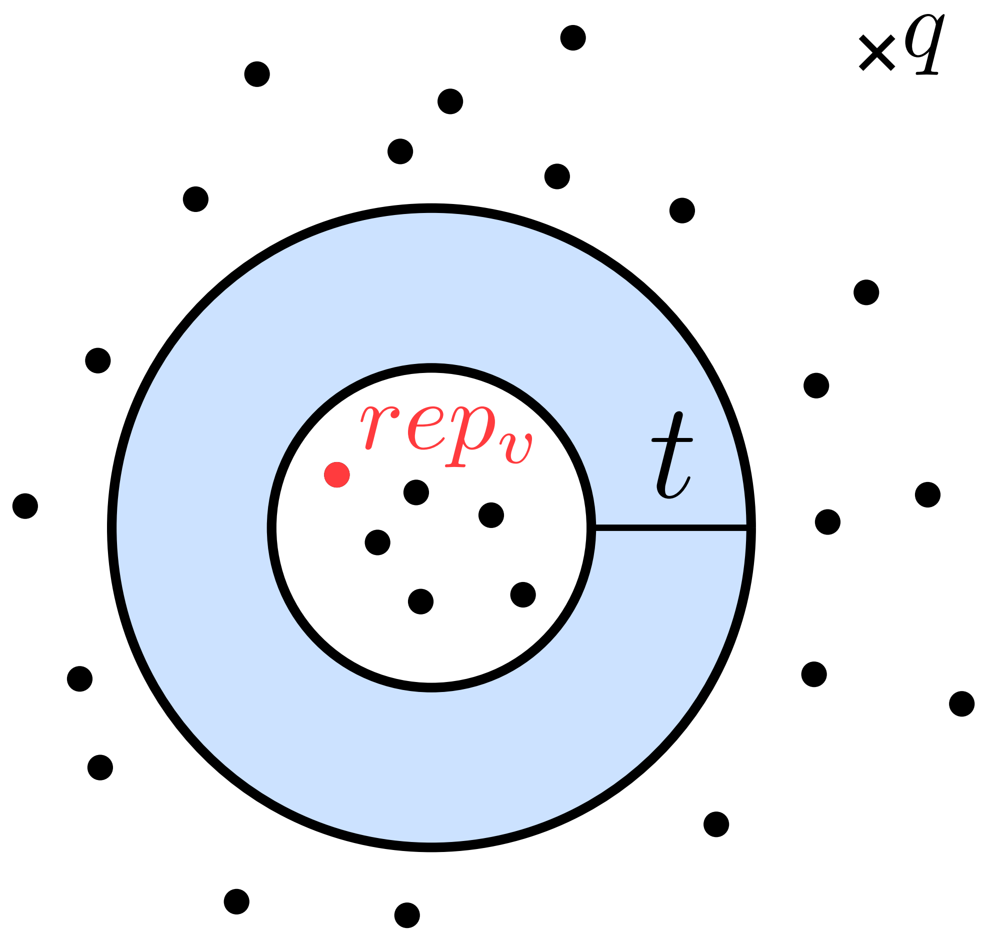

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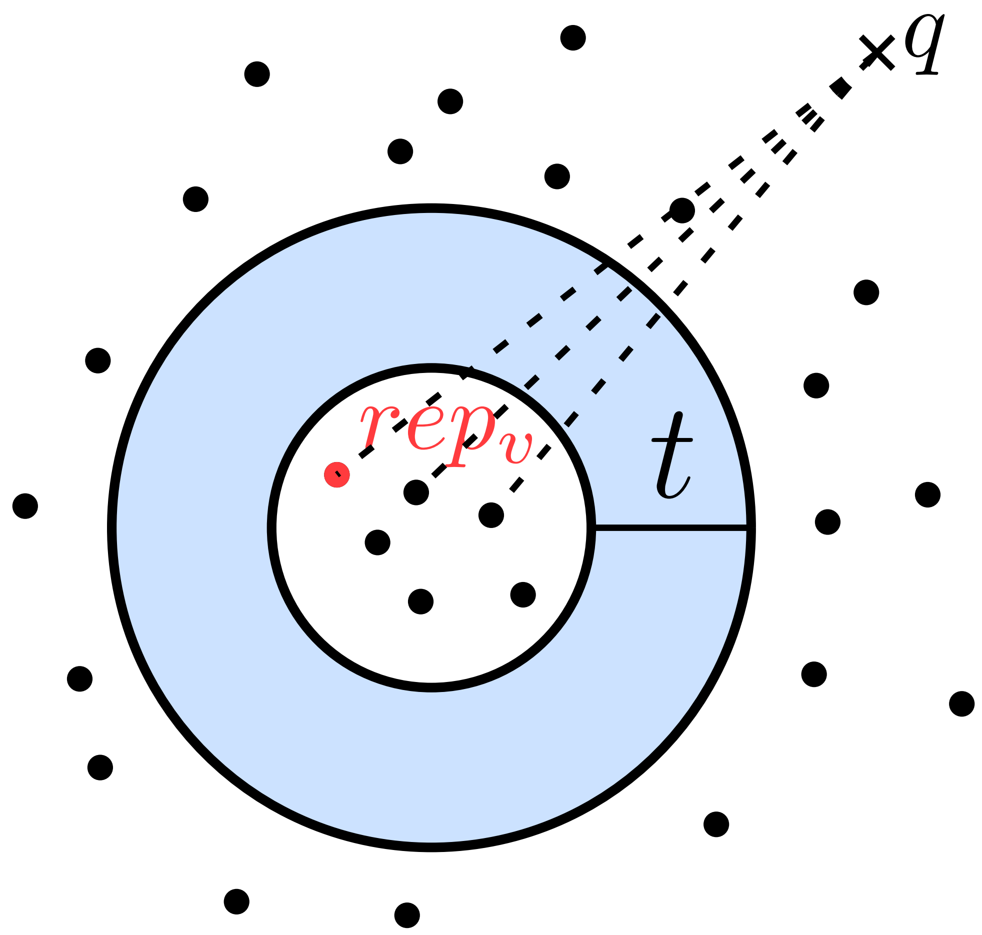

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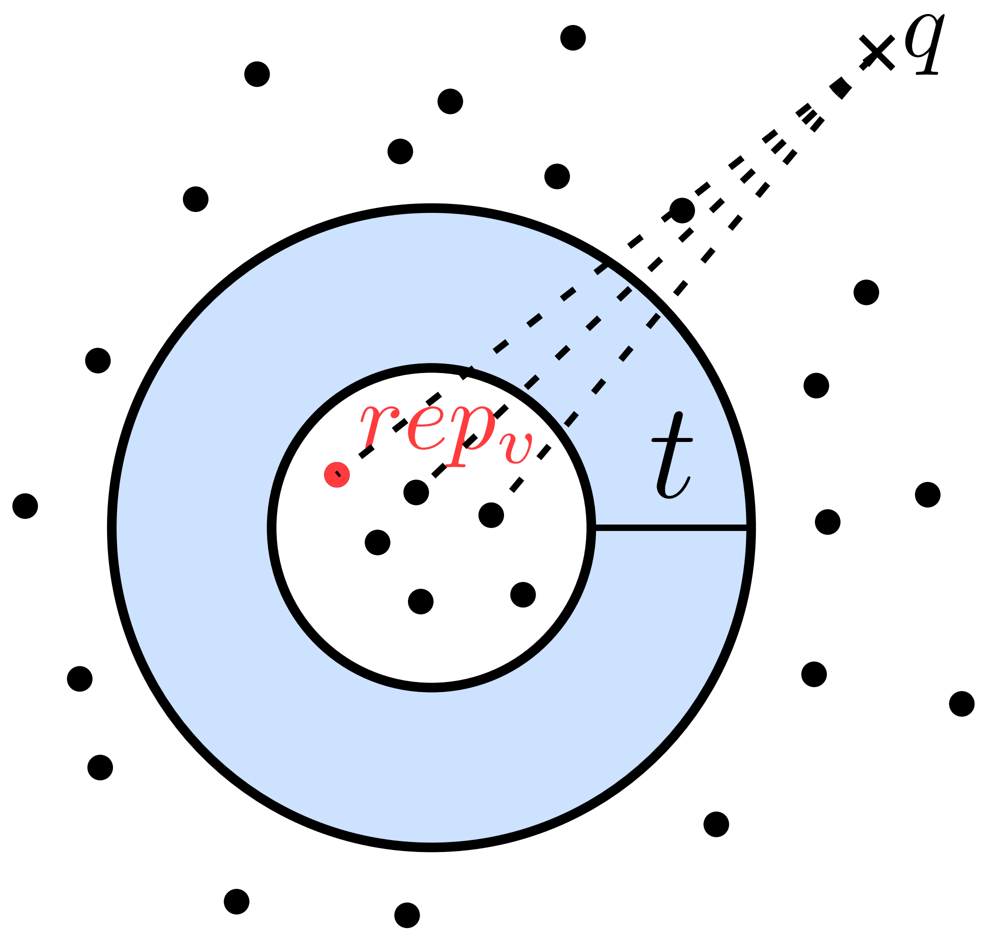

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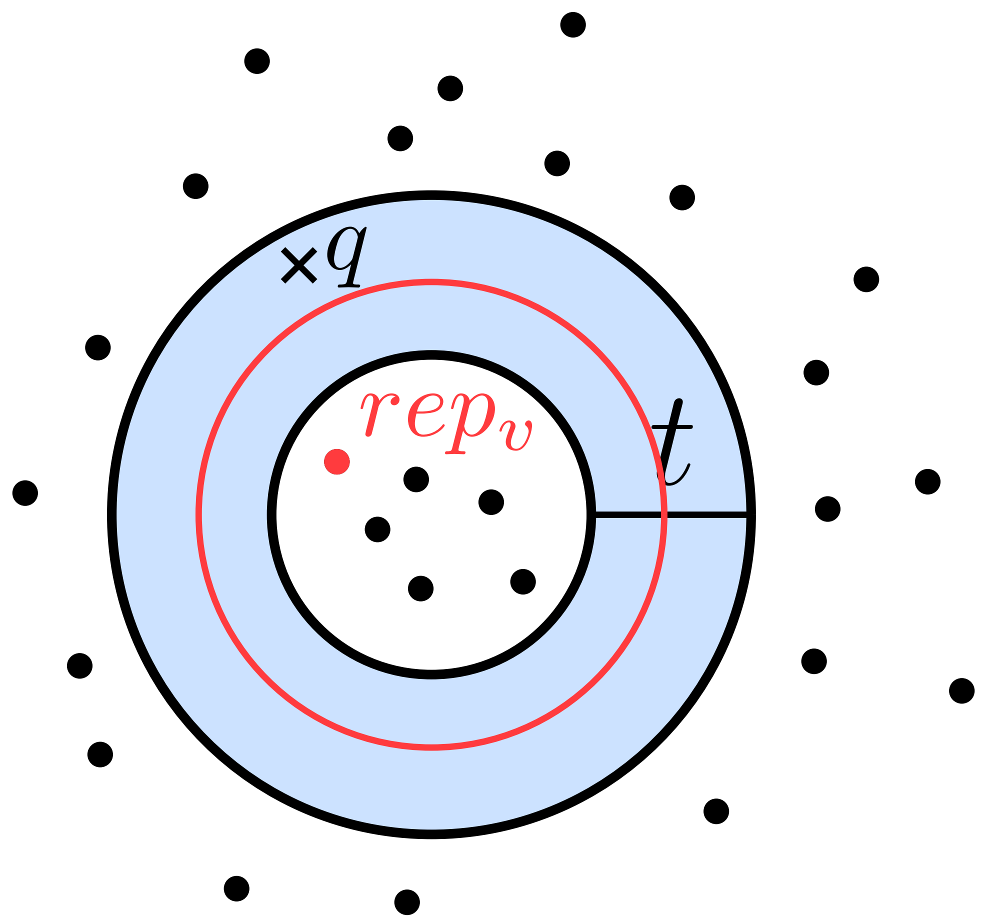

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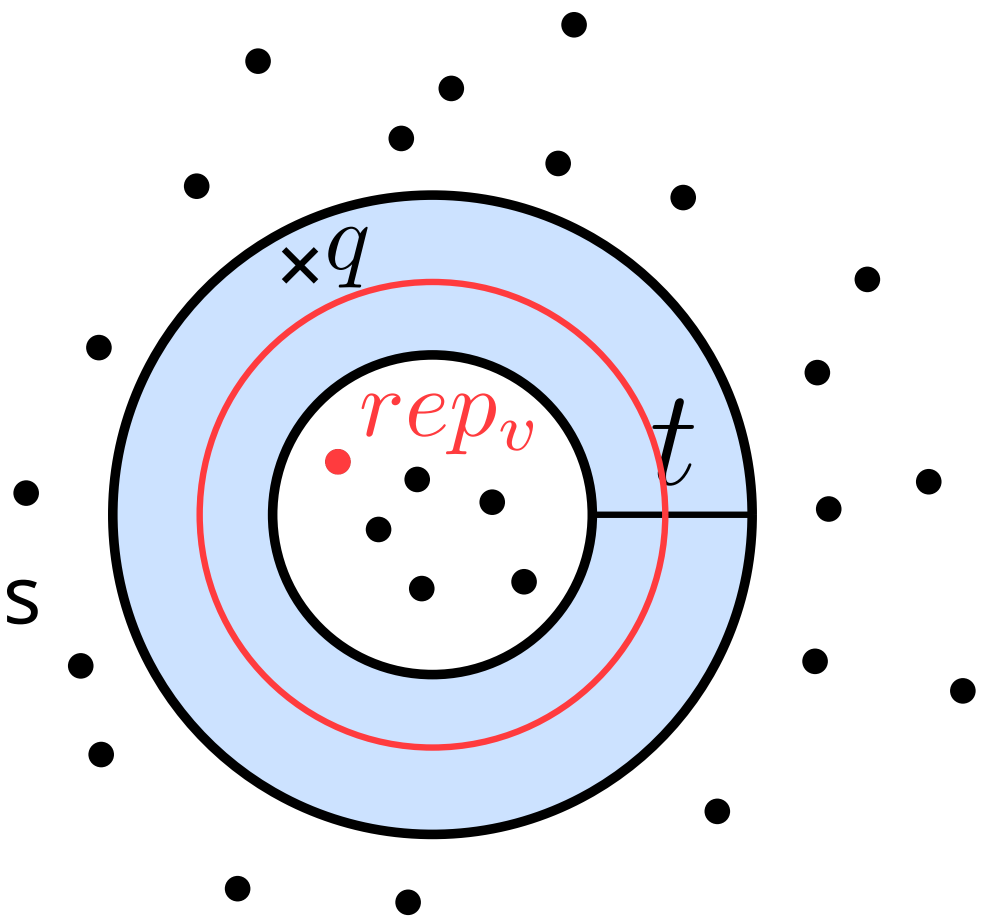
1.) or 2.) applies with slightly worse bounds



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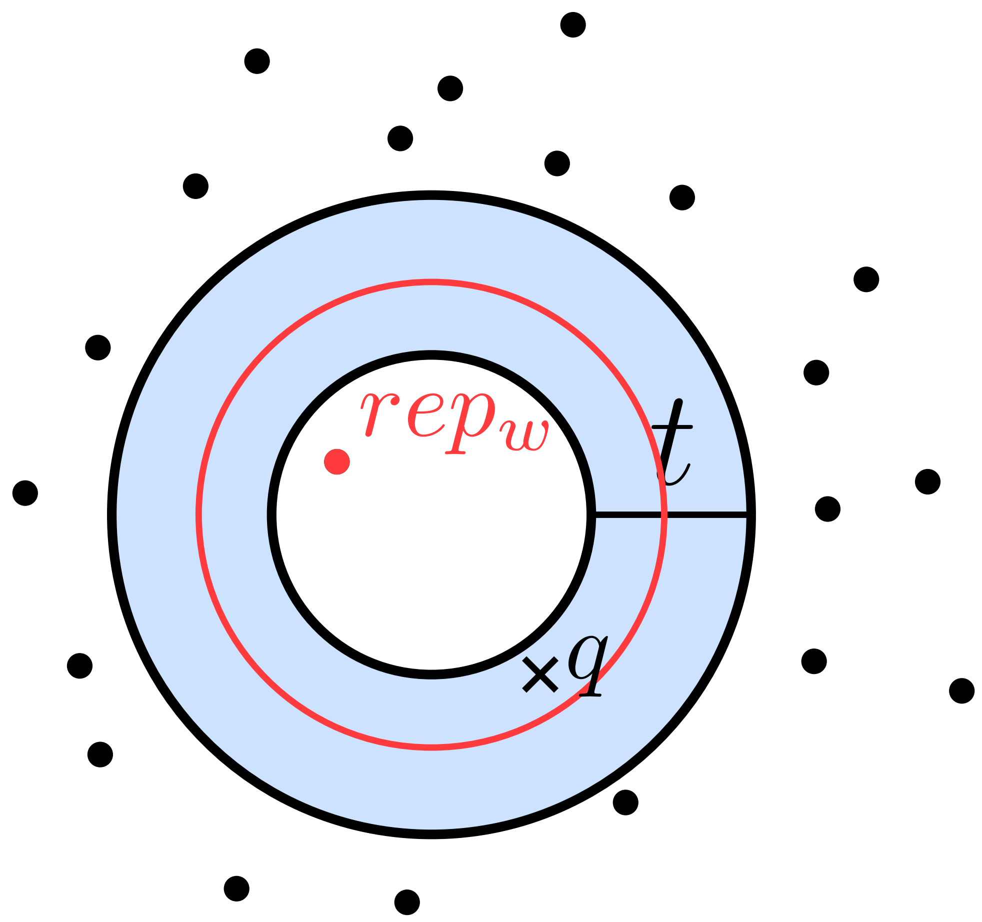

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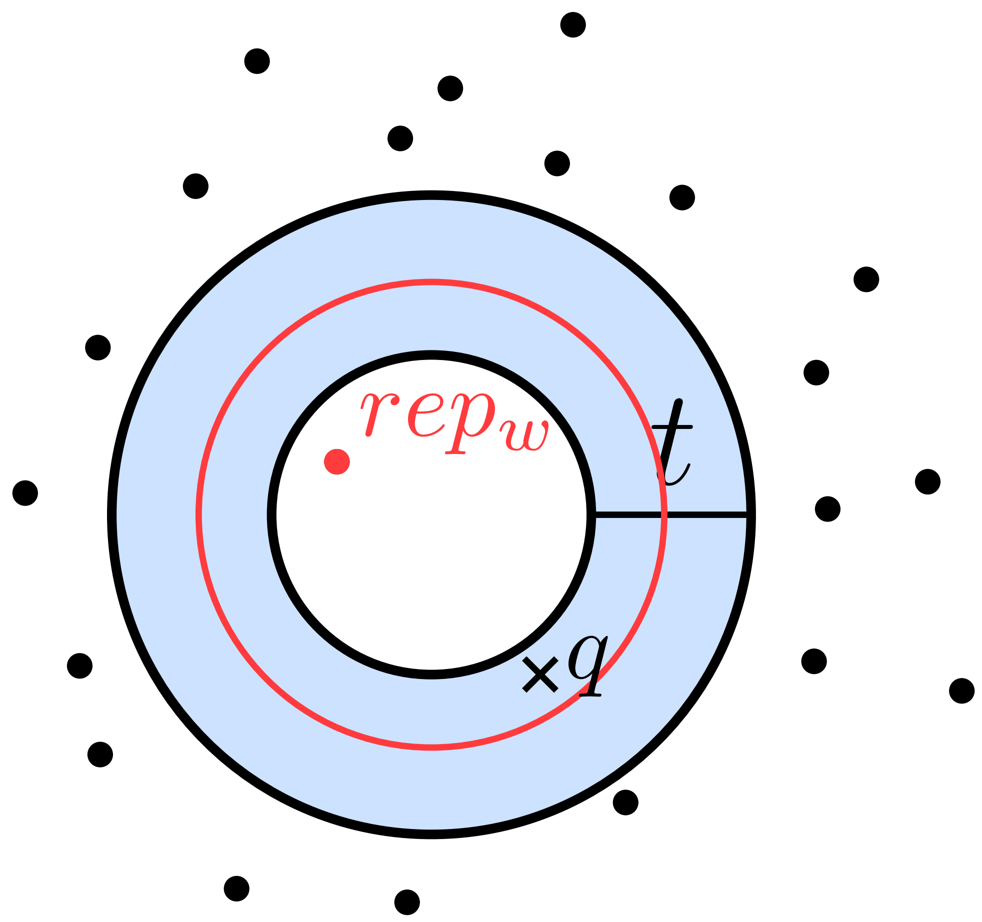

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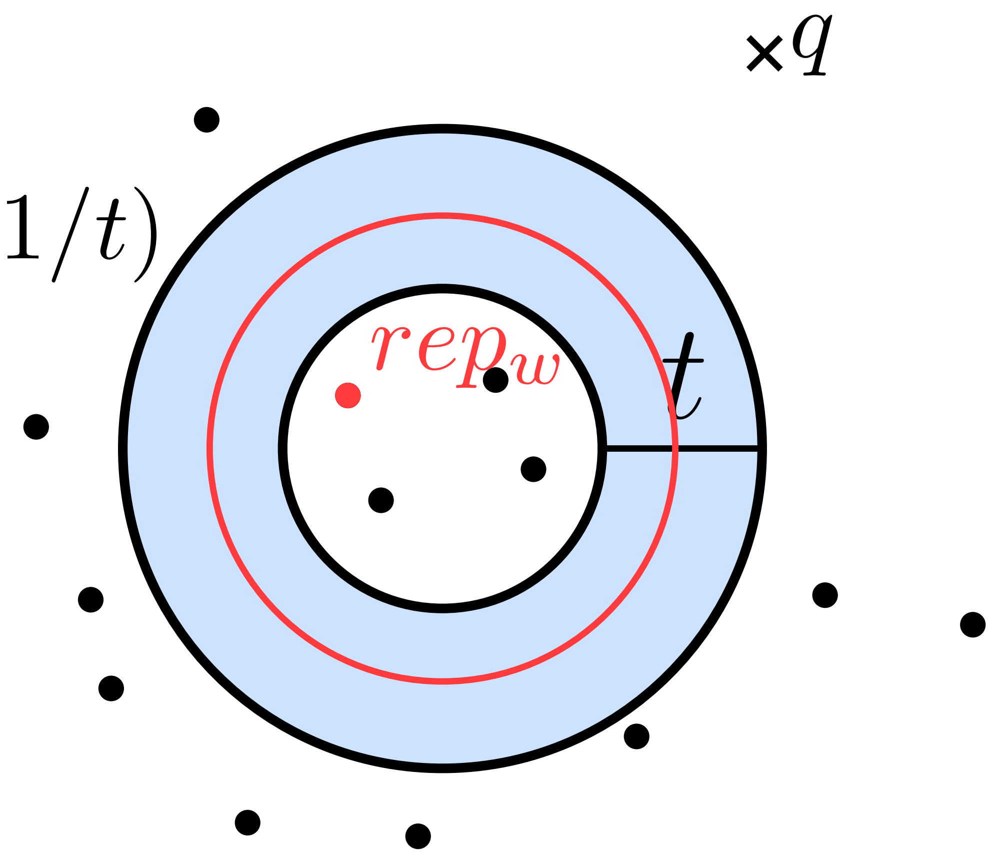

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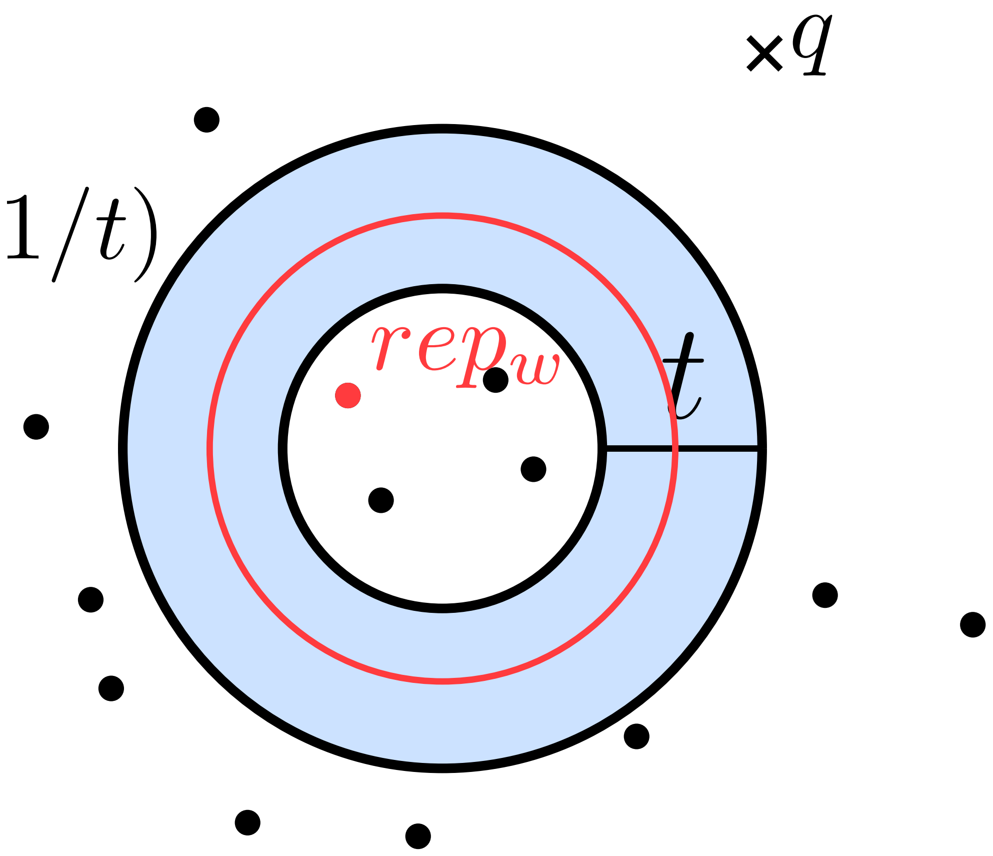
$$\frac{\|q-rep_w\|}{\|q-nn(q)\|} \le \frac{\|q-nn(q)\|+\|nn(q)-rep_w\|}{\|q-nn(q)\|}$$



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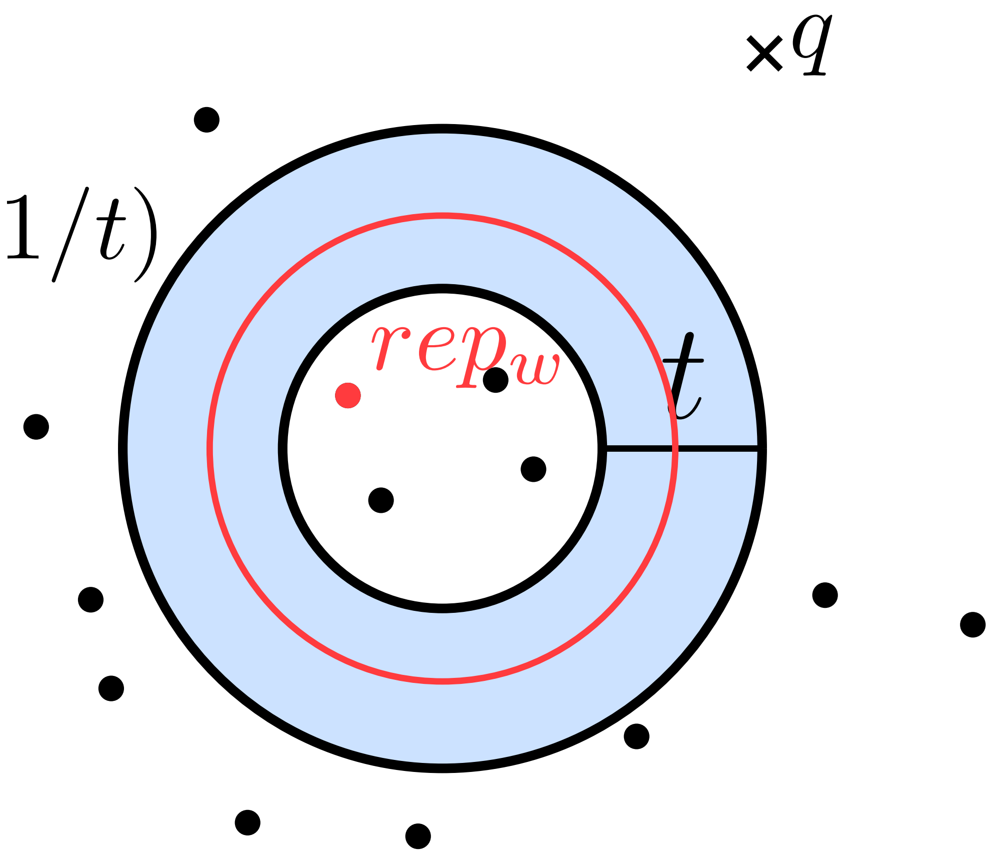
$$\leq 1+\frac{2r_w}{(t/2)r_w} = 1+4/t$$



Overview

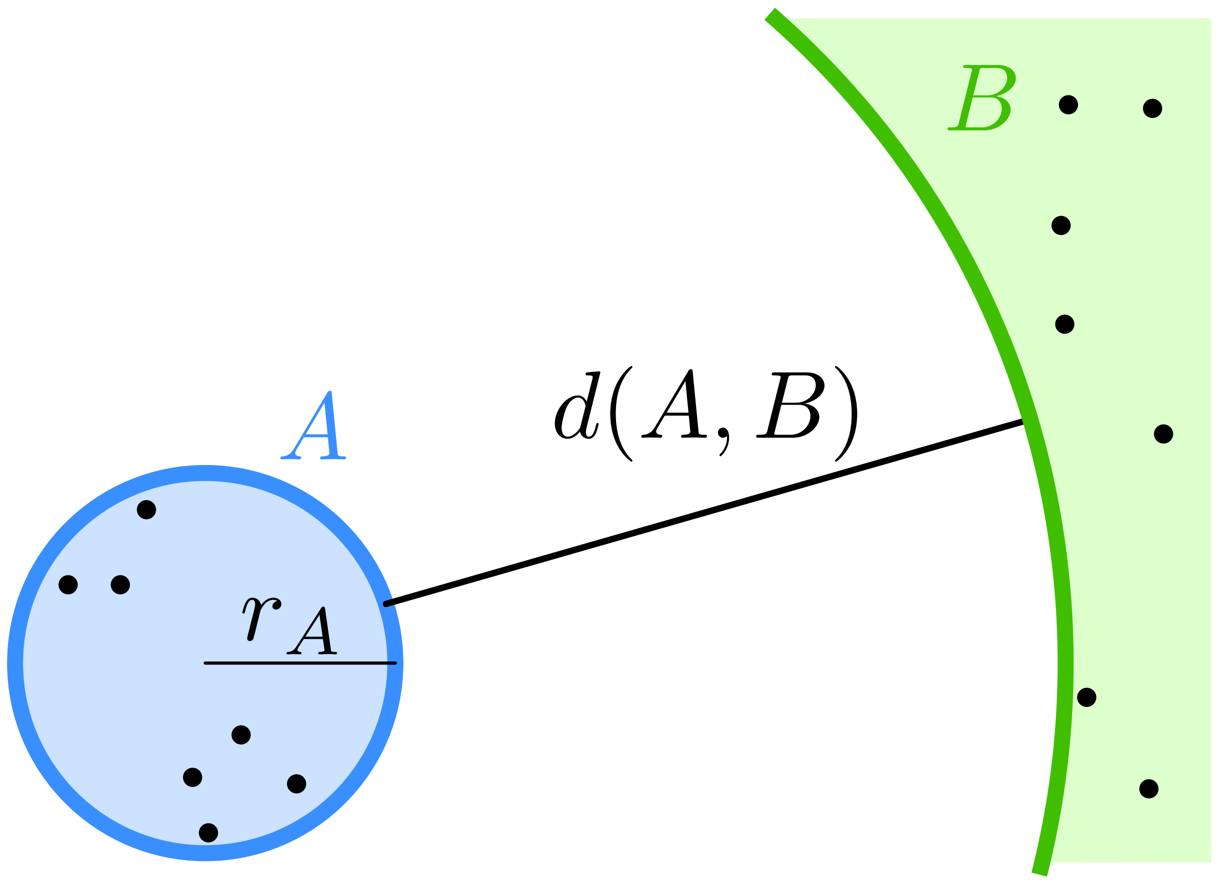
Semi-separated pair decomposition (SSPD)

Ring separator: n-semi-separated pair decomposition

Ring separator tree: $O(n)$-ANN

$(1/\varepsilon)$-semi-separated pair decomposition – brief sketch

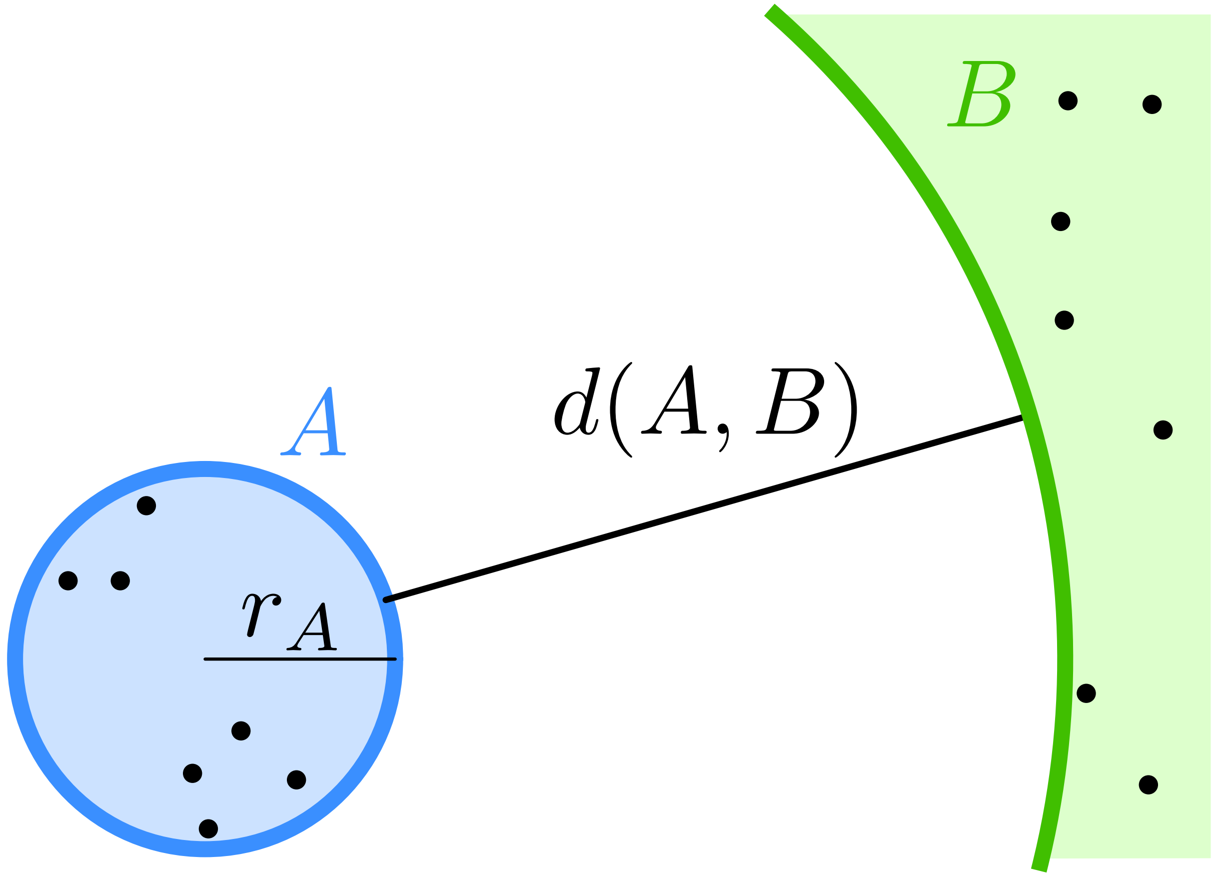
$(1/\varepsilon)$-Semi-Separated Pairs



ring separator:

- ball $b = b(p, r)$ containing $\geq n/c_1$ points
- no point in $b(p, r(1 + 1/n)) \setminus b$
- $\geq n/c_2$ points outside $b(p, 2r)$

$(1/\varepsilon)$-Semi-Separated Pairs



$$P_{in} = P \cap b, P_{far} = P \setminus b(p, 2r/\varepsilon)$$
$$P_{out} = P \setminus (P_{in} \cup P_{far})$$

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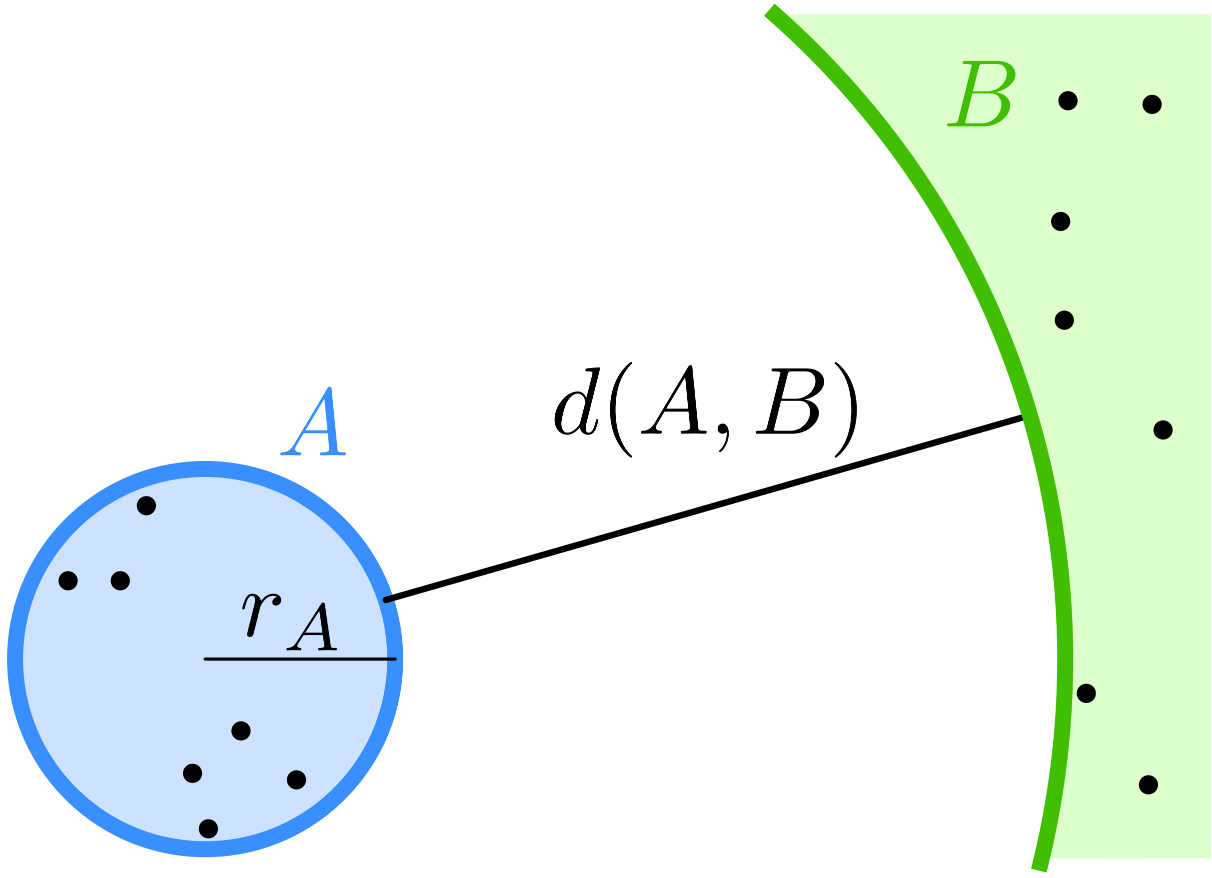

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2. recurse on P_{in}, P_{in}

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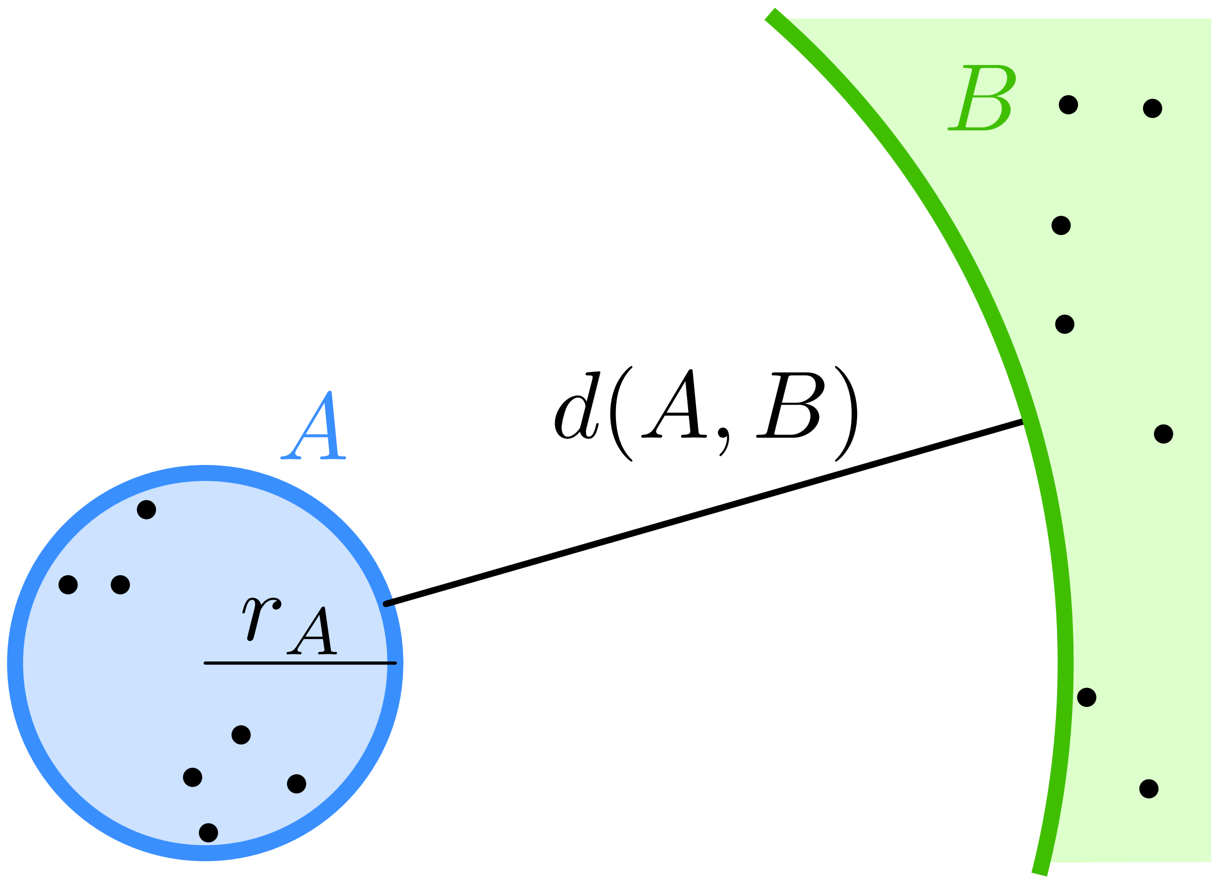

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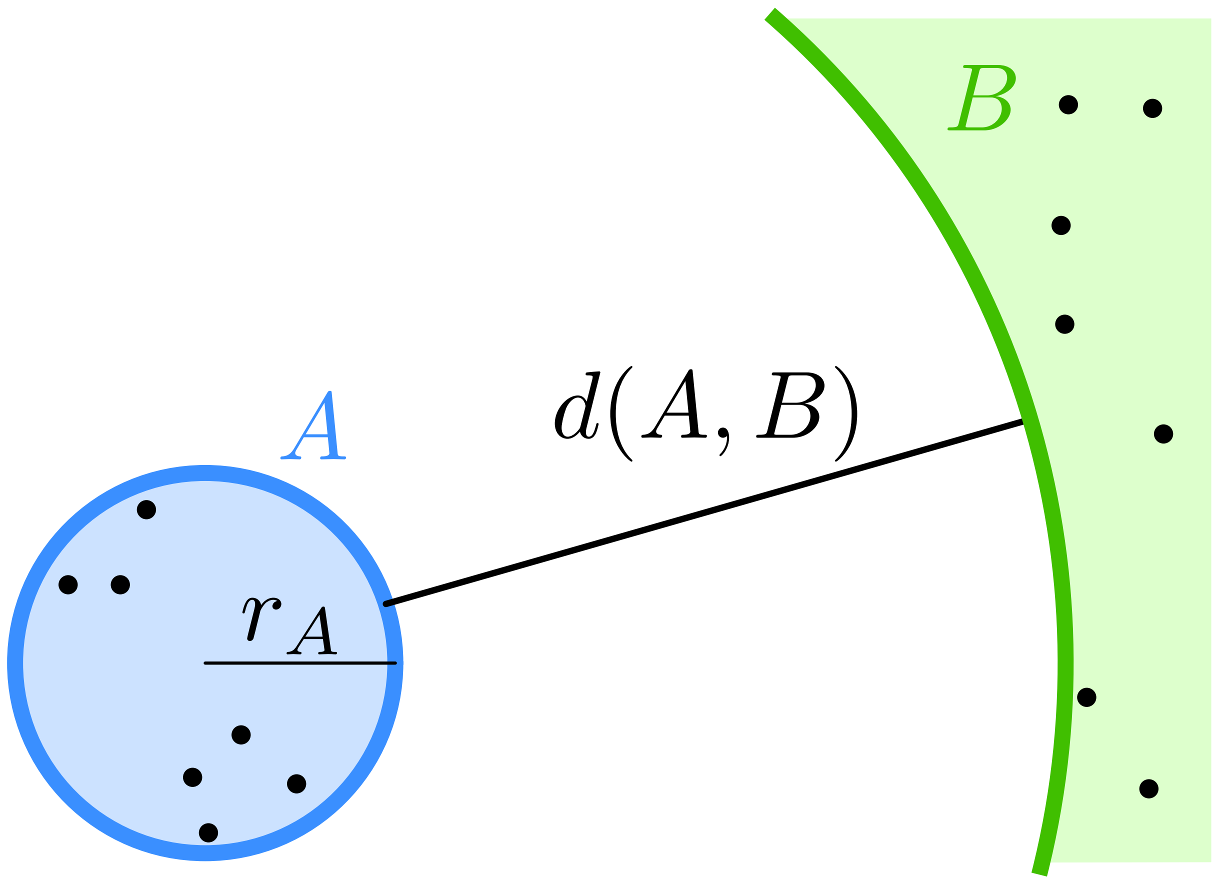

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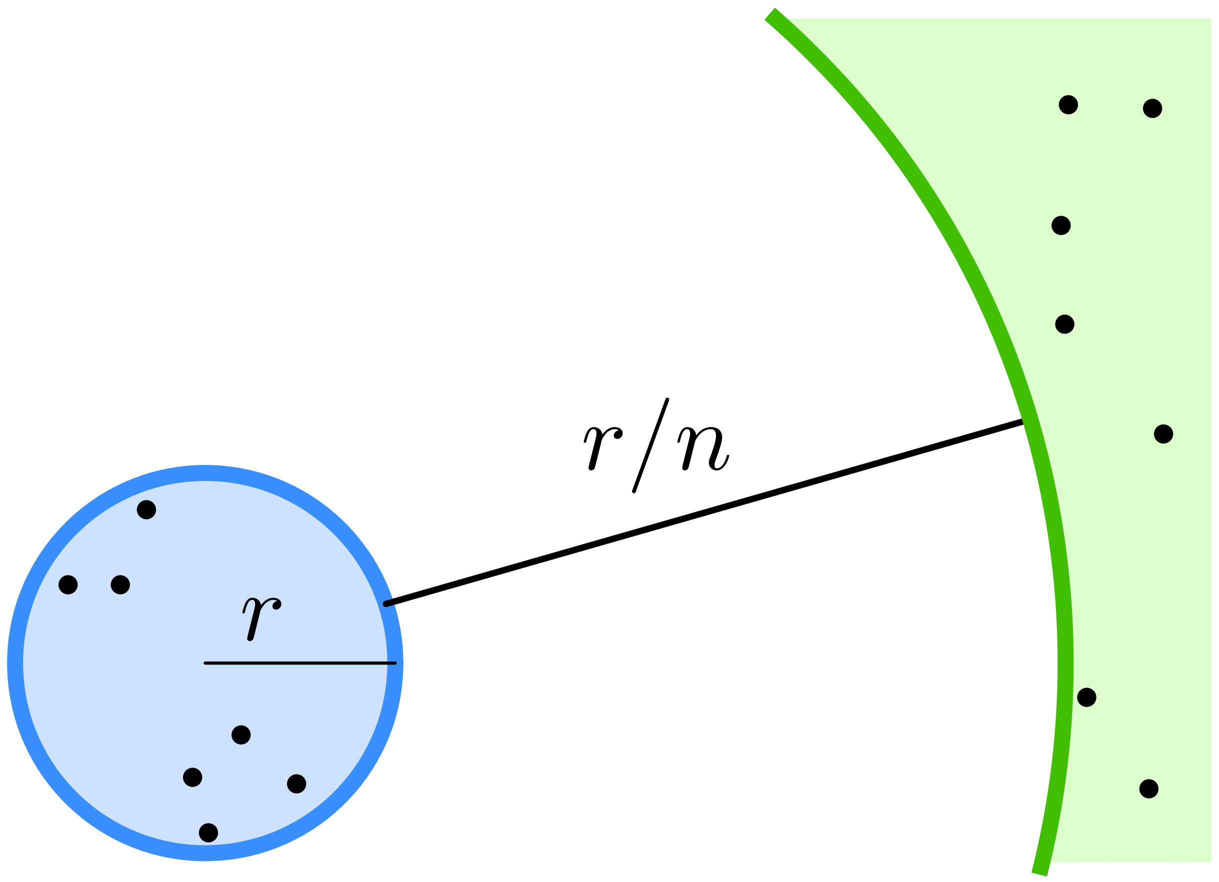
Dealing with P_{in}, P_{out} (sketch)



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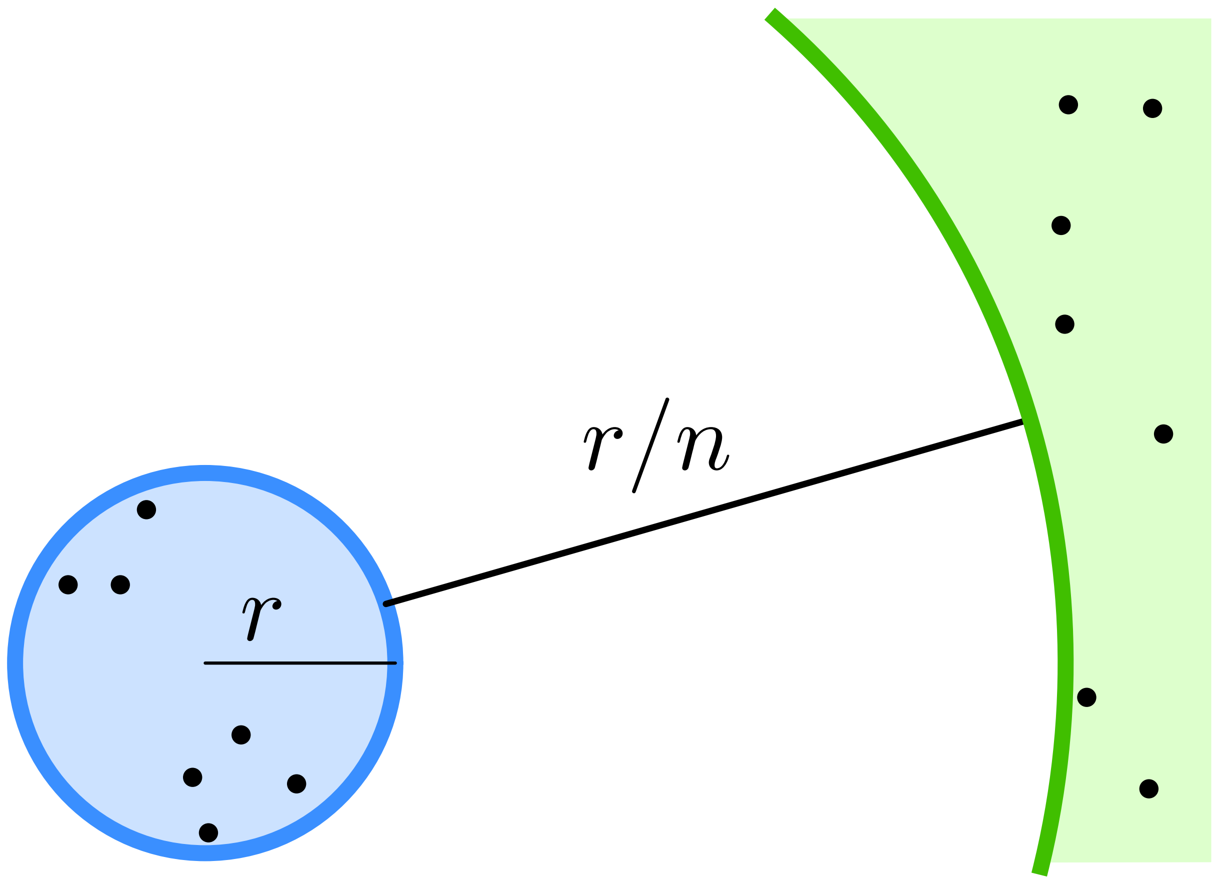

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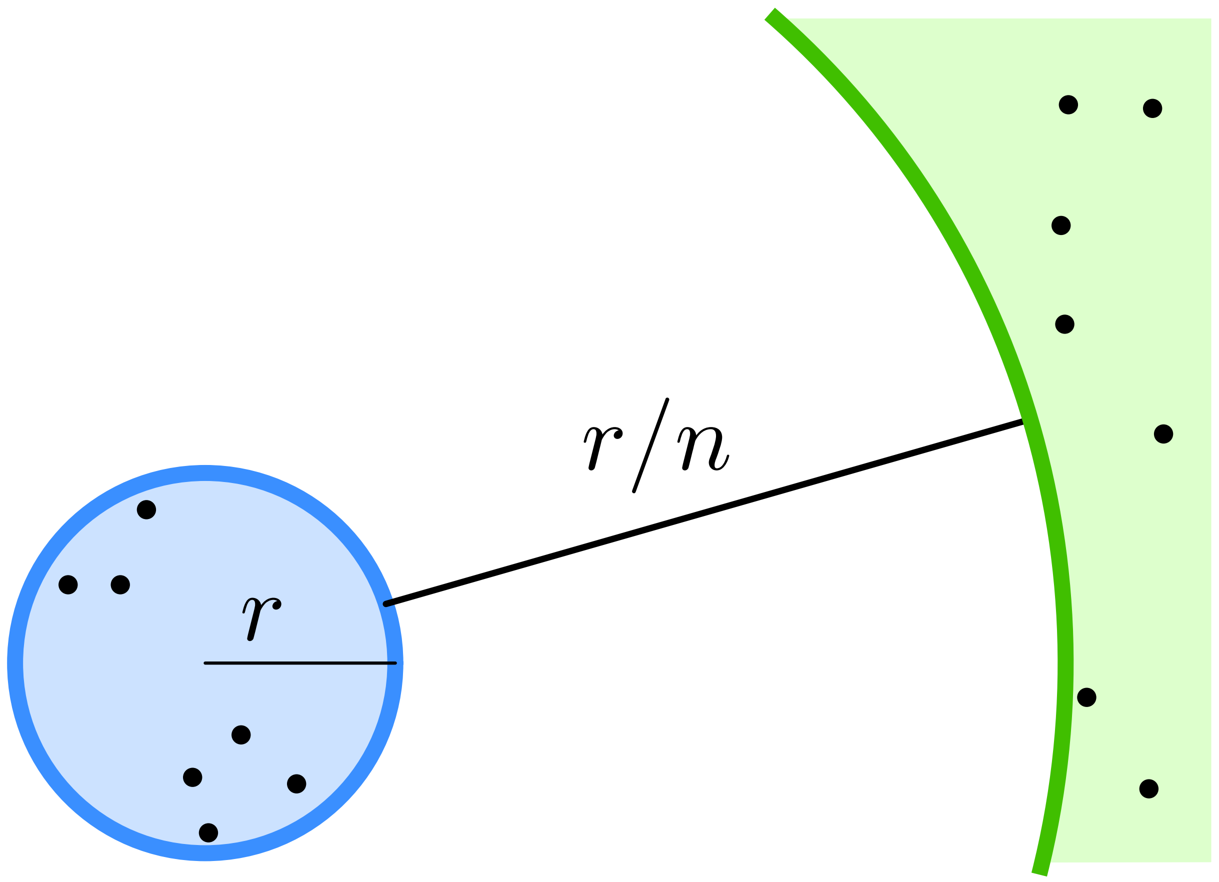

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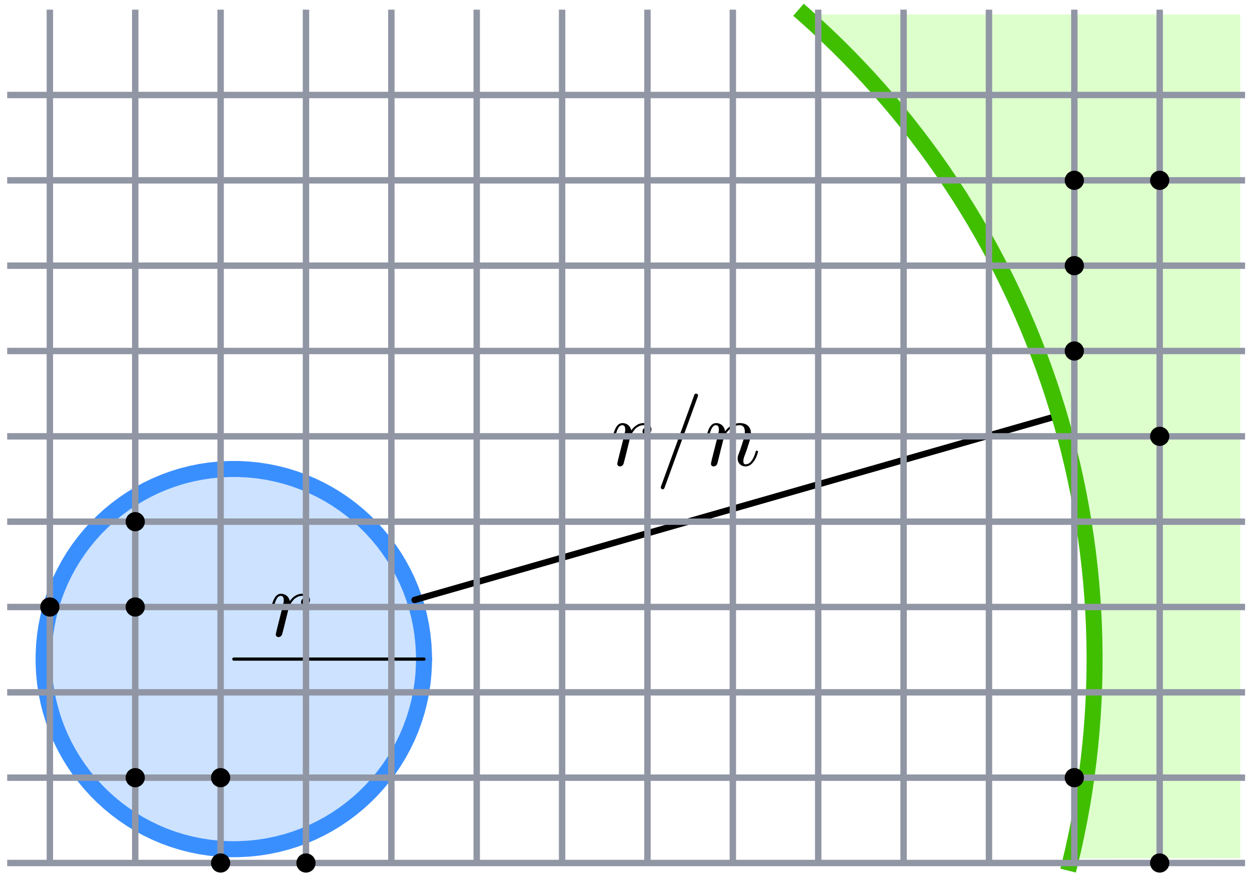

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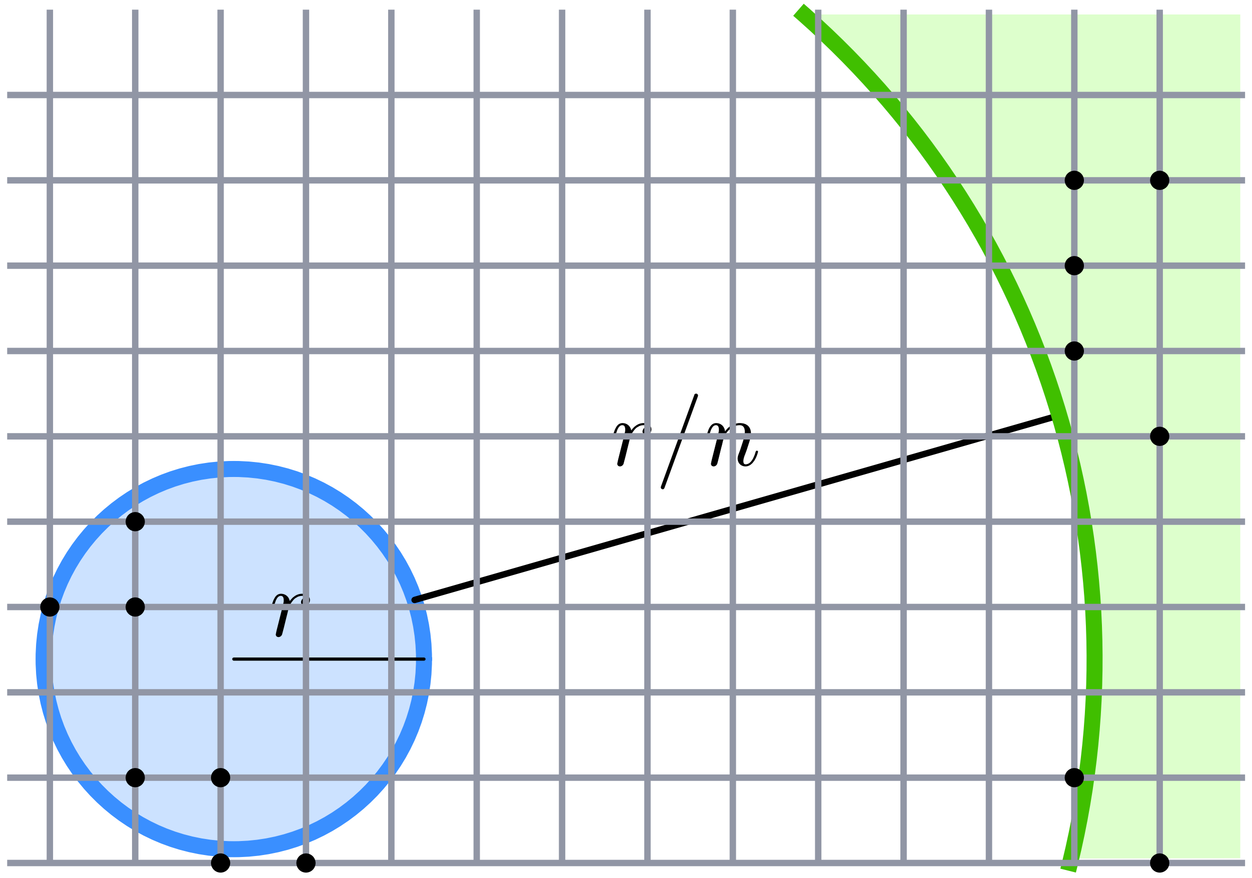

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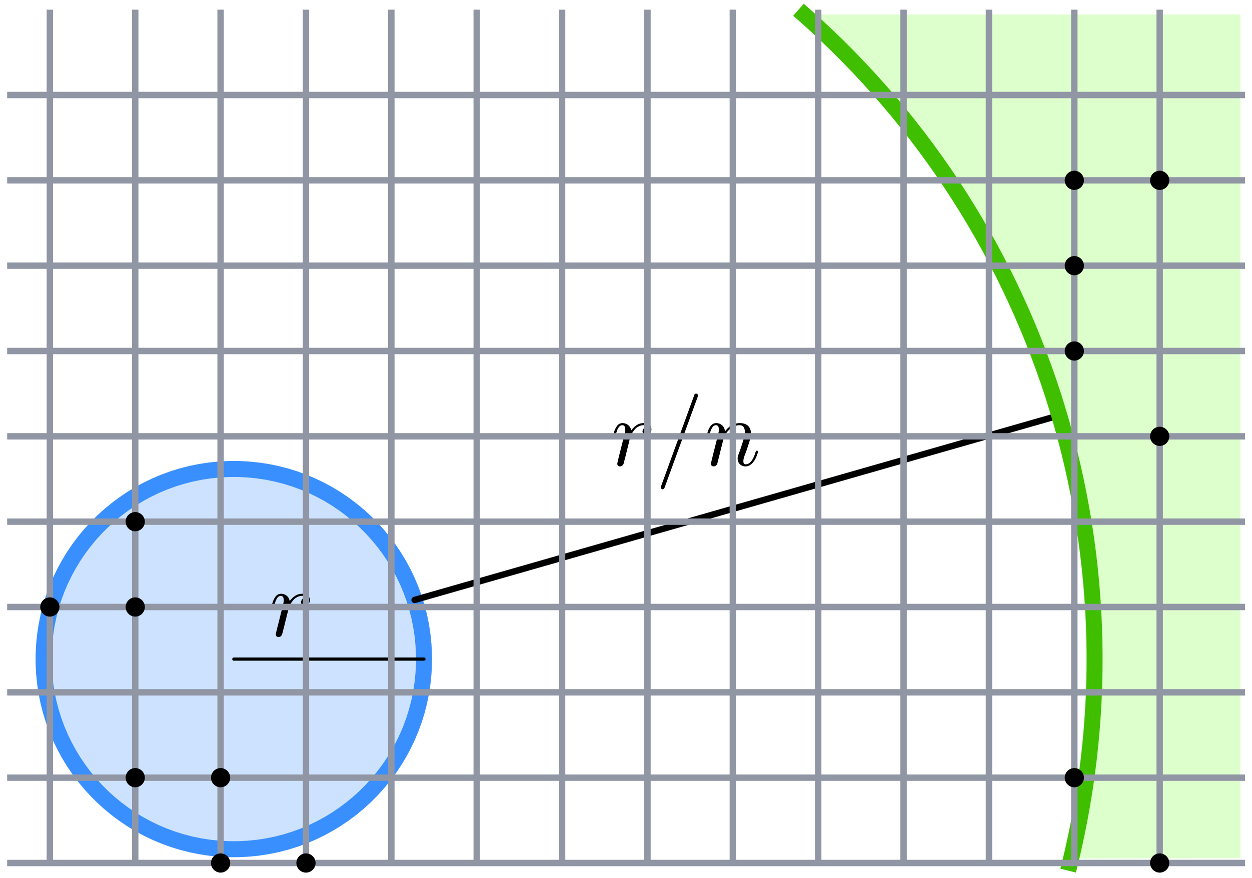

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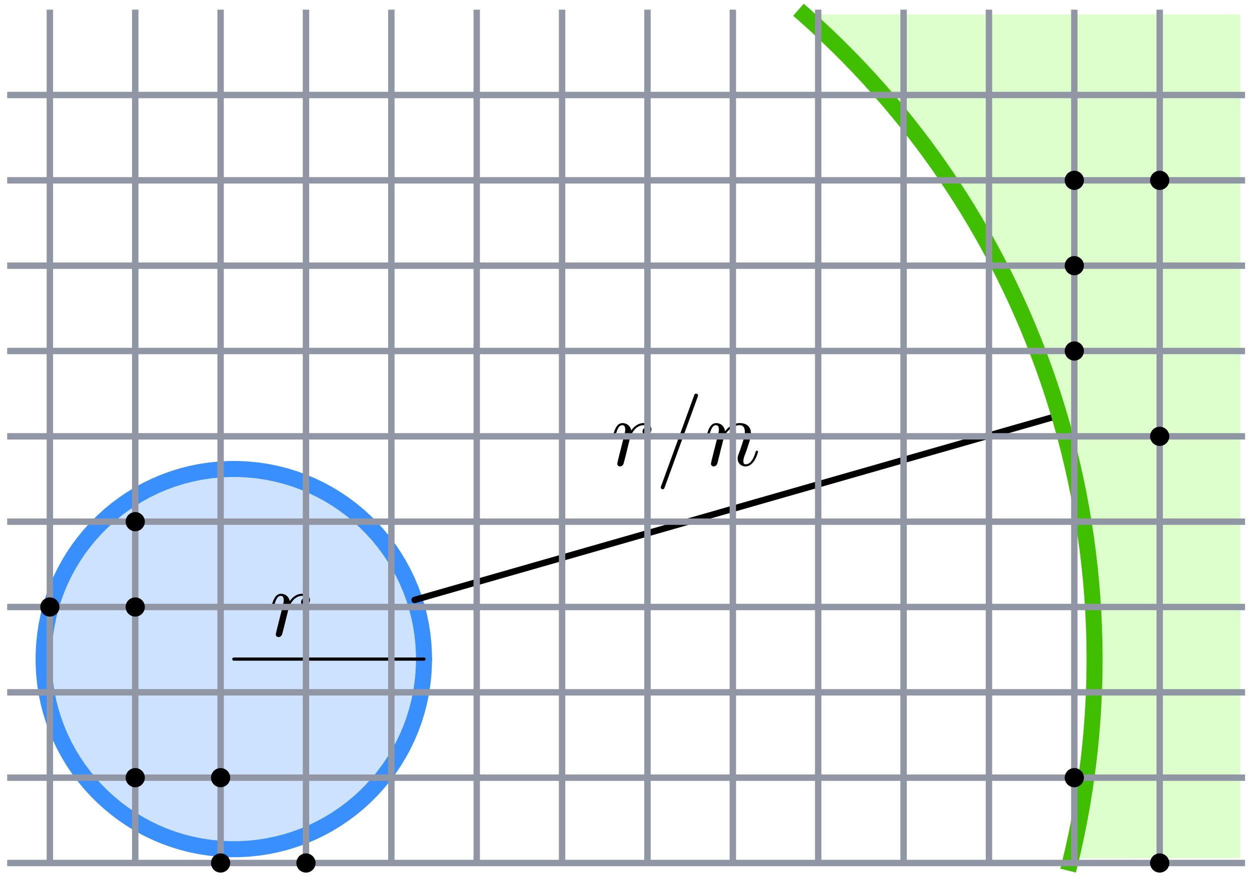

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  $\rightarrow$ # pairs = $O(n \log n)$ (low weight)

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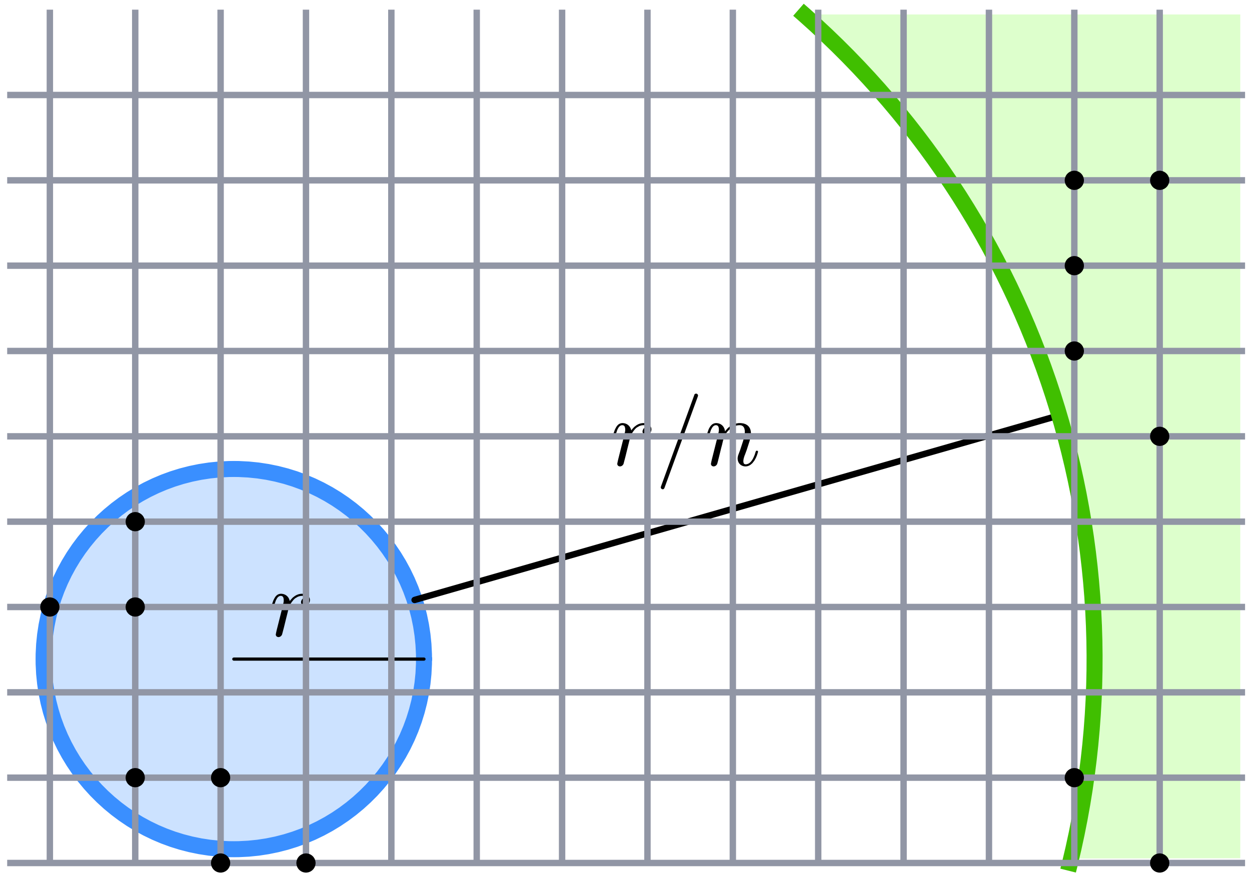

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- grid size: $O(n/\varepsilon^2)$
- levels in quadtree: $O(\log(n/\varepsilon^2))$
  $\rightarrow$ # pairs = $O(n \log n)$ (low weight)
- snapping: since $\ell \geq r/n$ distances between $p \in P_{in}$ and $q \in P_{out}$ approximately stay the same.

Summary

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$$\min(r_A, r_B) \leq \varepsilon d(A, B)$$

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$(1/\varepsilon)$-semi-separated pair decomposition

n-ring separator + snap to grid + WSPD