Combinatorial Discrepancy

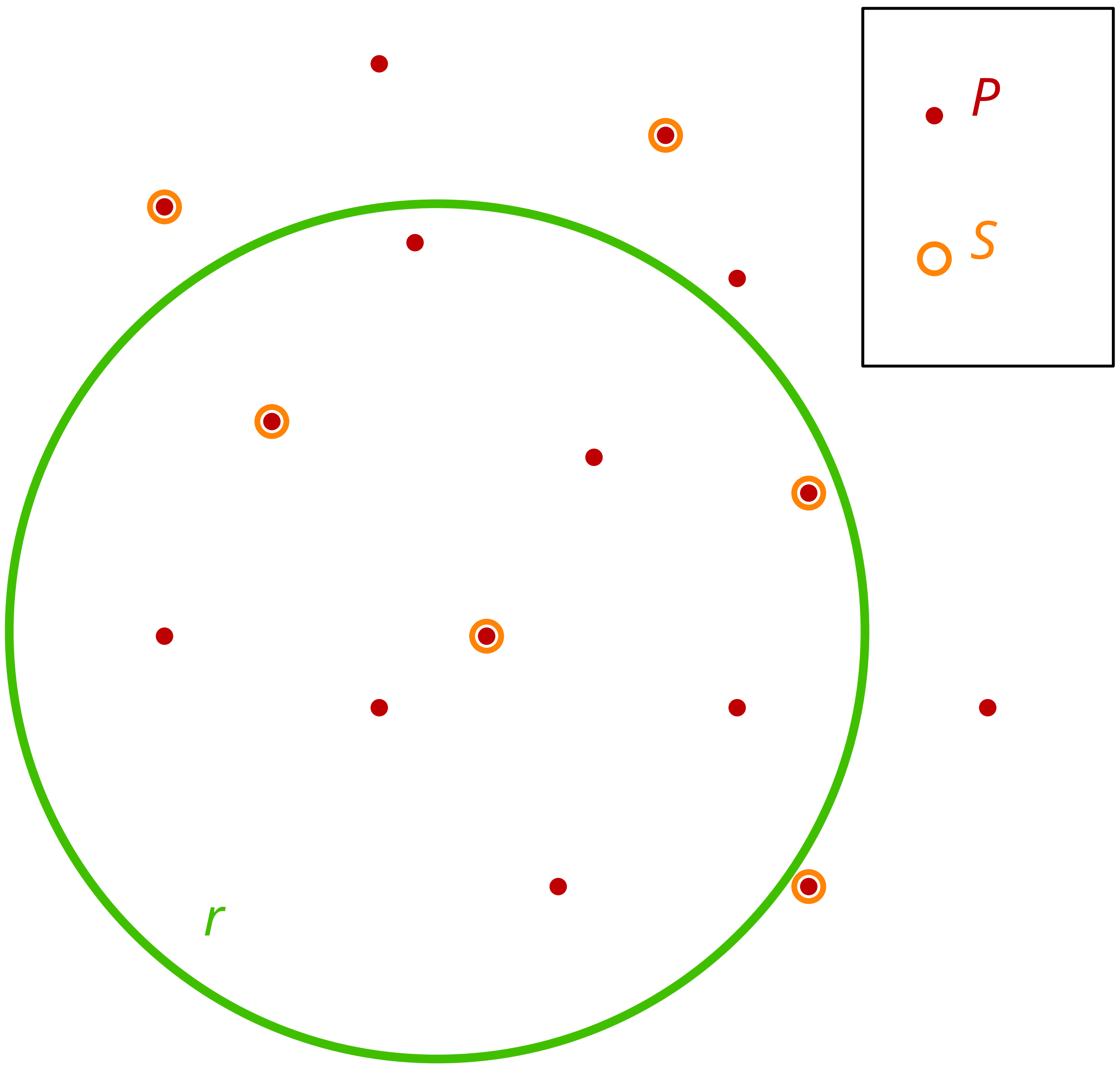
sampling using discrepancy

computing spanning trees with low stabbing number via reweighting

ε-samples

Measure: $\mu(r) = \frac{|r \cap P|}{|P|}$

Estimate: $\hat{\mu}(r) = \frac{|r \cap S|}{|S|}$



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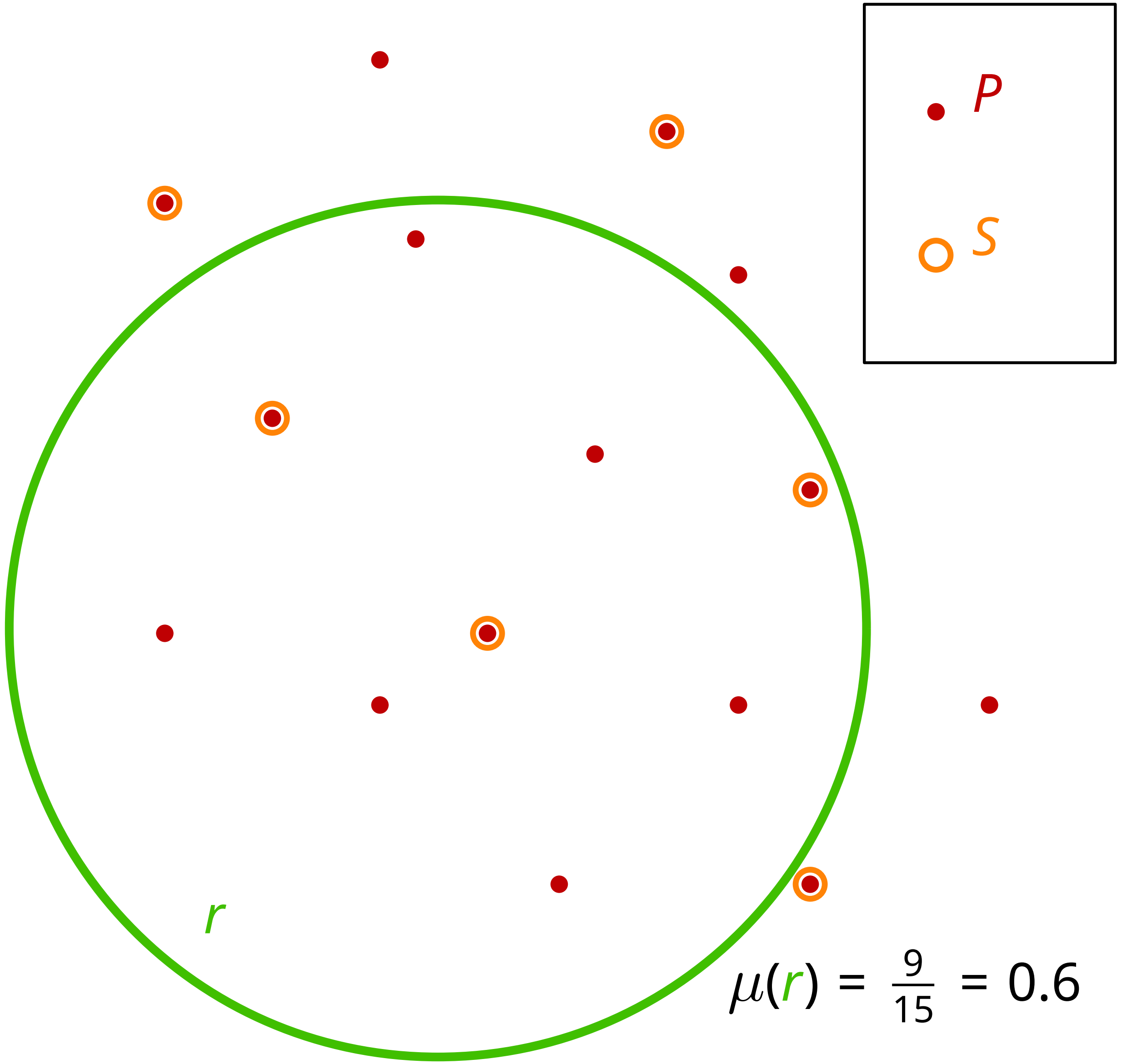

$\mu(r) = \frac{9}{15} = 0.6$

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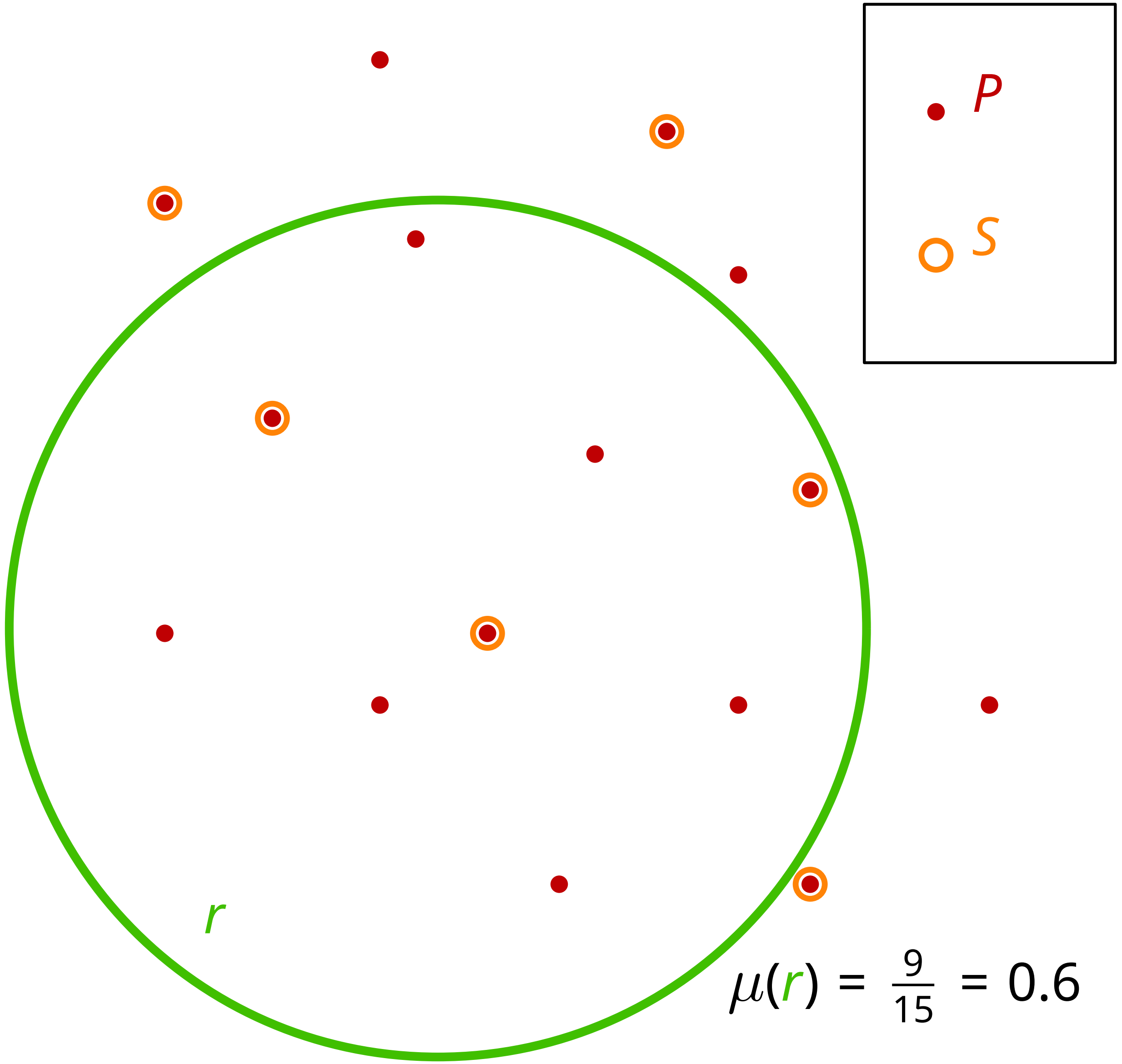

$\mu(r) = \frac{9}{15} = 0.6$

$\hat{\mu}(r) = \frac{3}{6} = 0.5$

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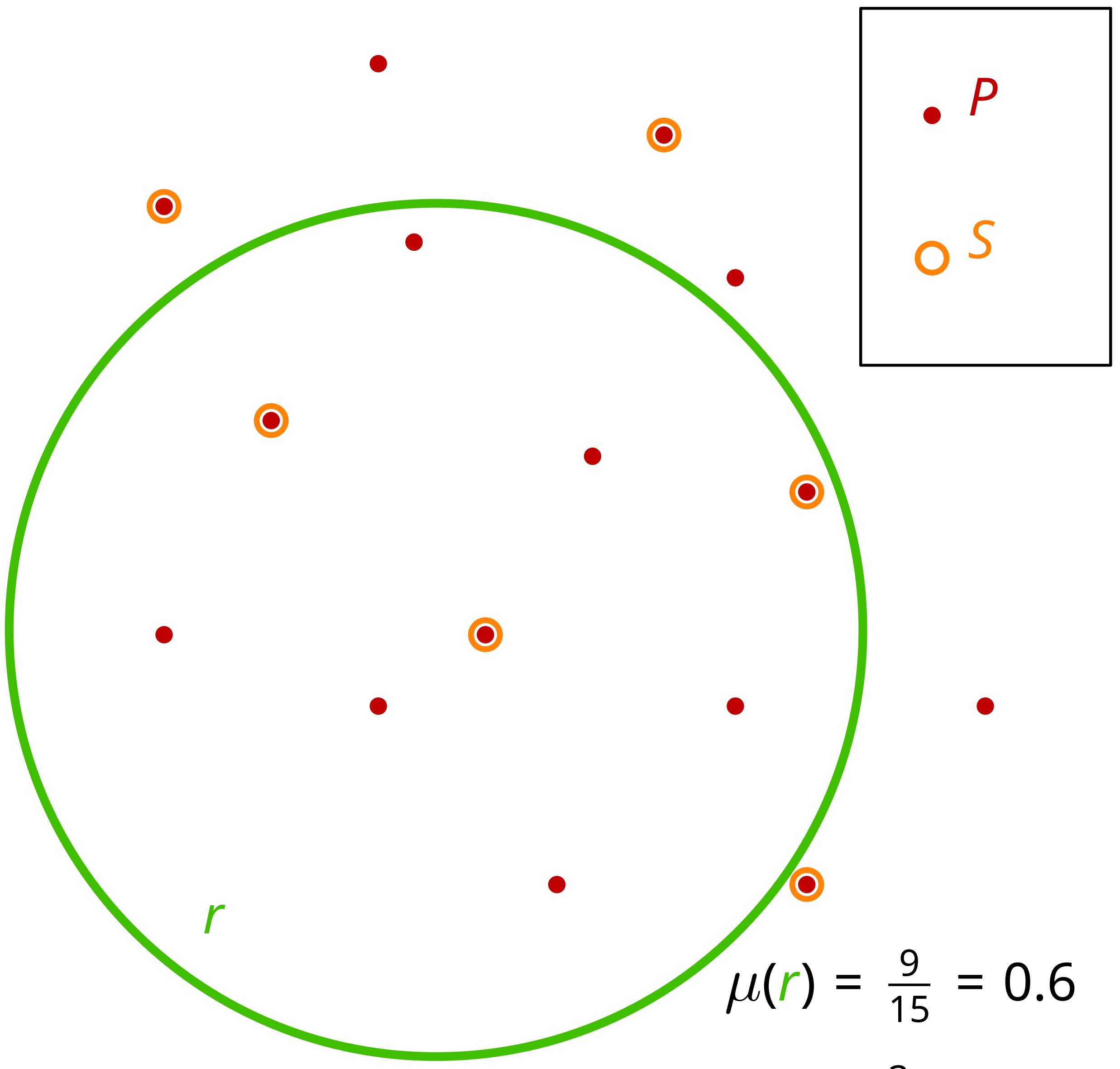
ε-sample S:

for all $r \in \mathcal{R}$ and any $0 \leq \varepsilon \leq 1$

$\left|\mu(r) - \hat{\mu}(r)\right| \leq \varepsilon$



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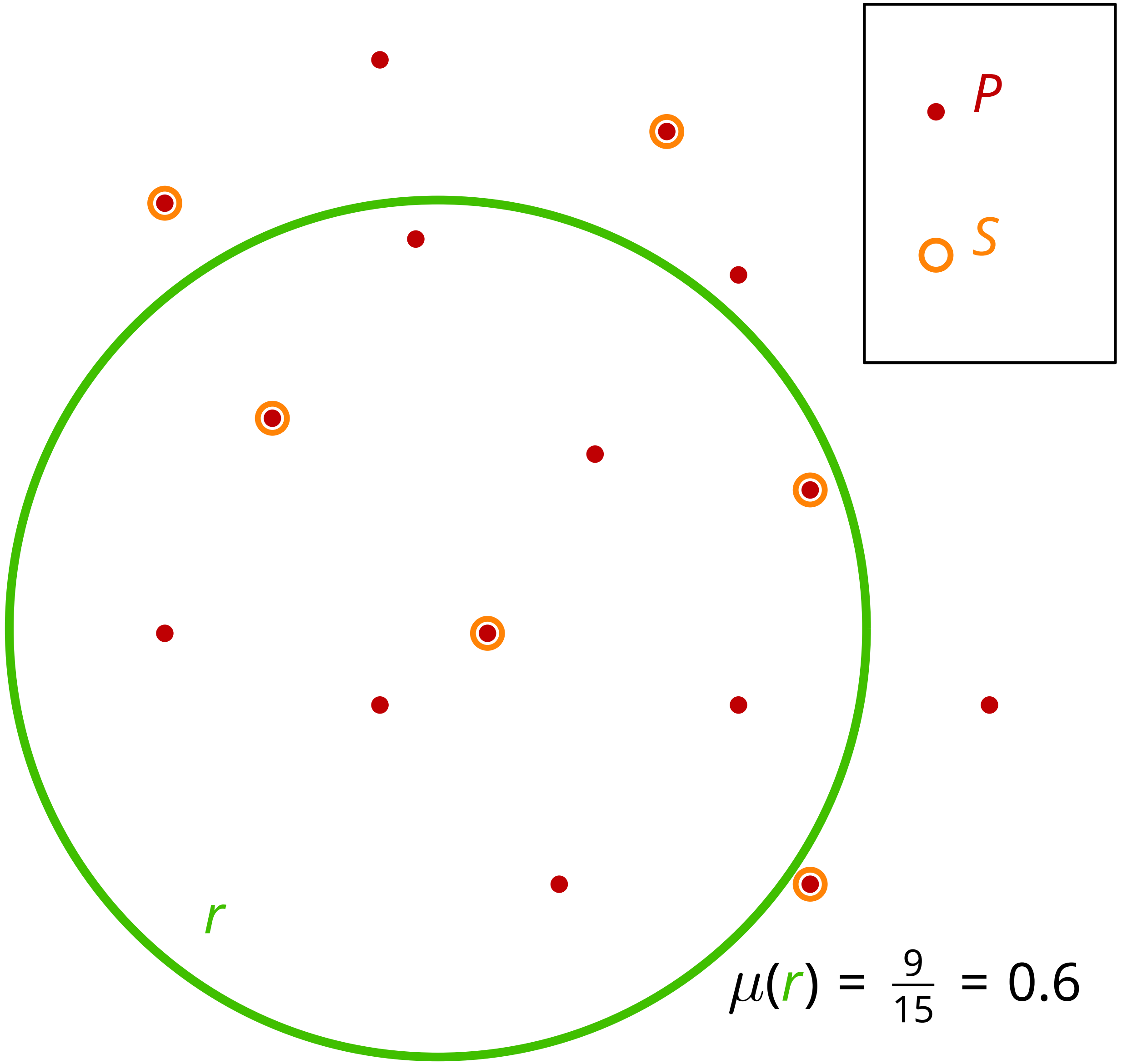
ε-sample theorem: For constant $p > 0$ and VC-dim. a random sample of size $O(1/\varepsilon^2)$ is an ε-sample with probability p.



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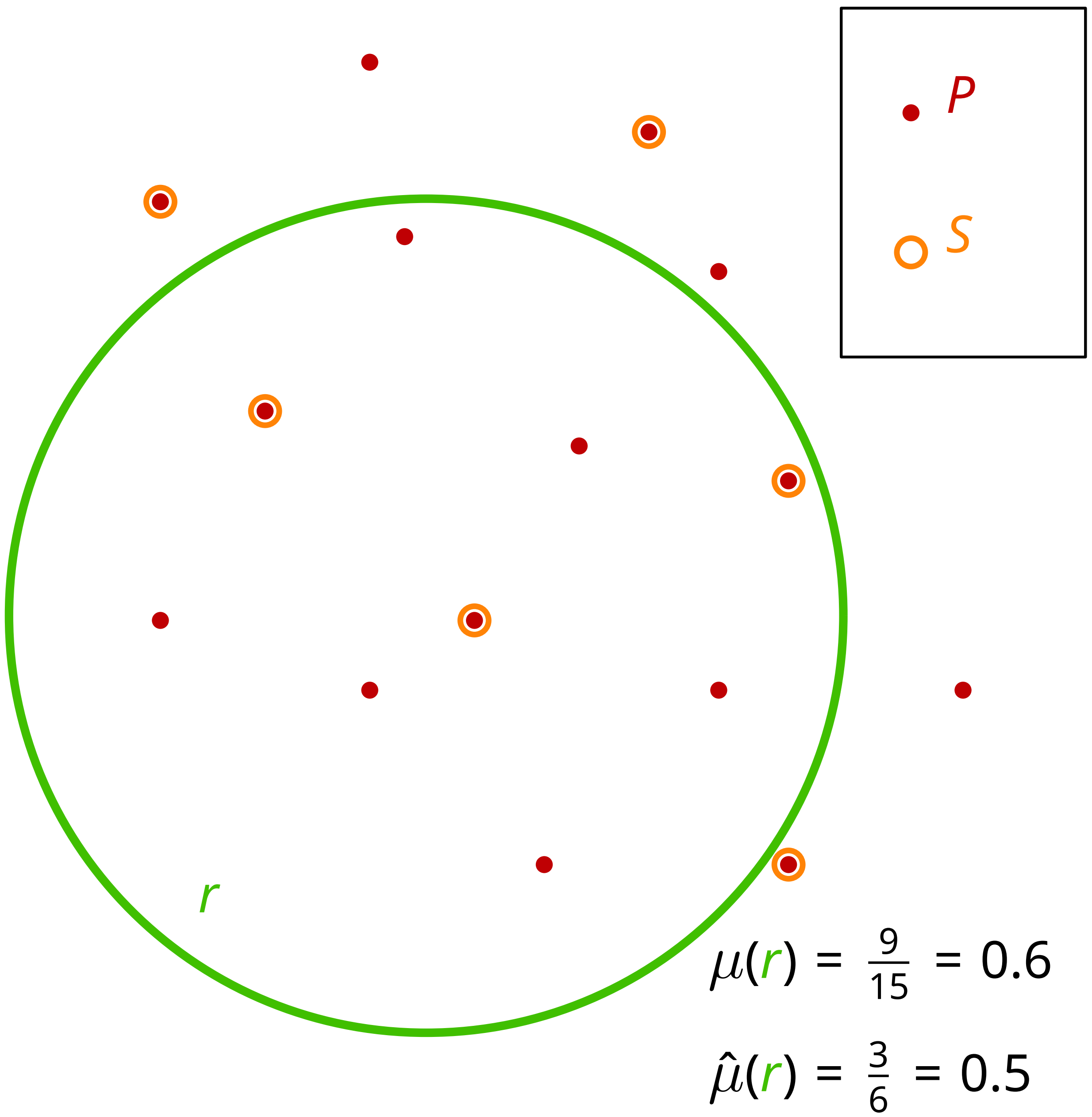
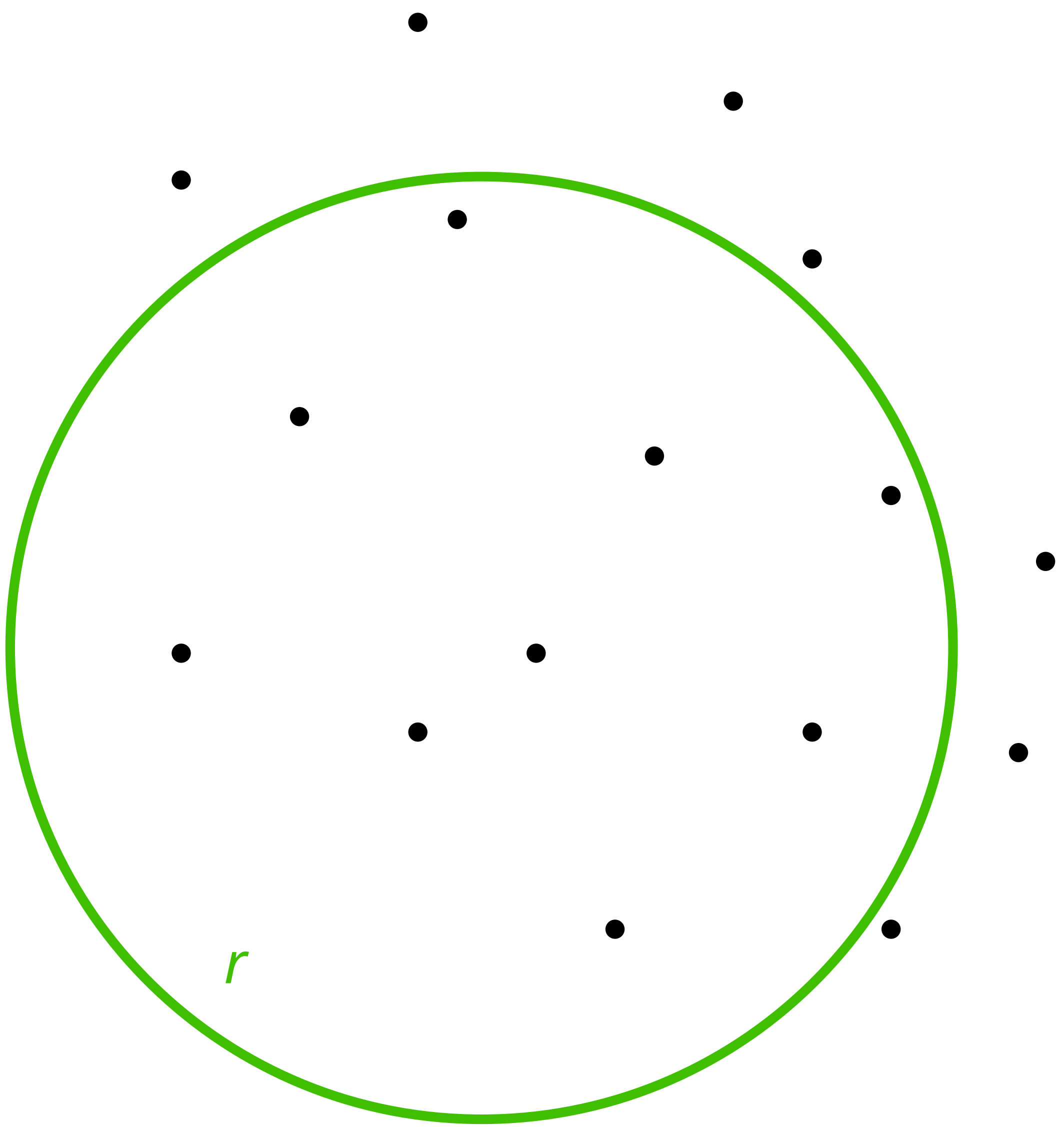
Smaller size? Deterministic construction?

Via discrepancy!



Discrepancy

Color P in two colors: '1' (red) and '-1' (blue)



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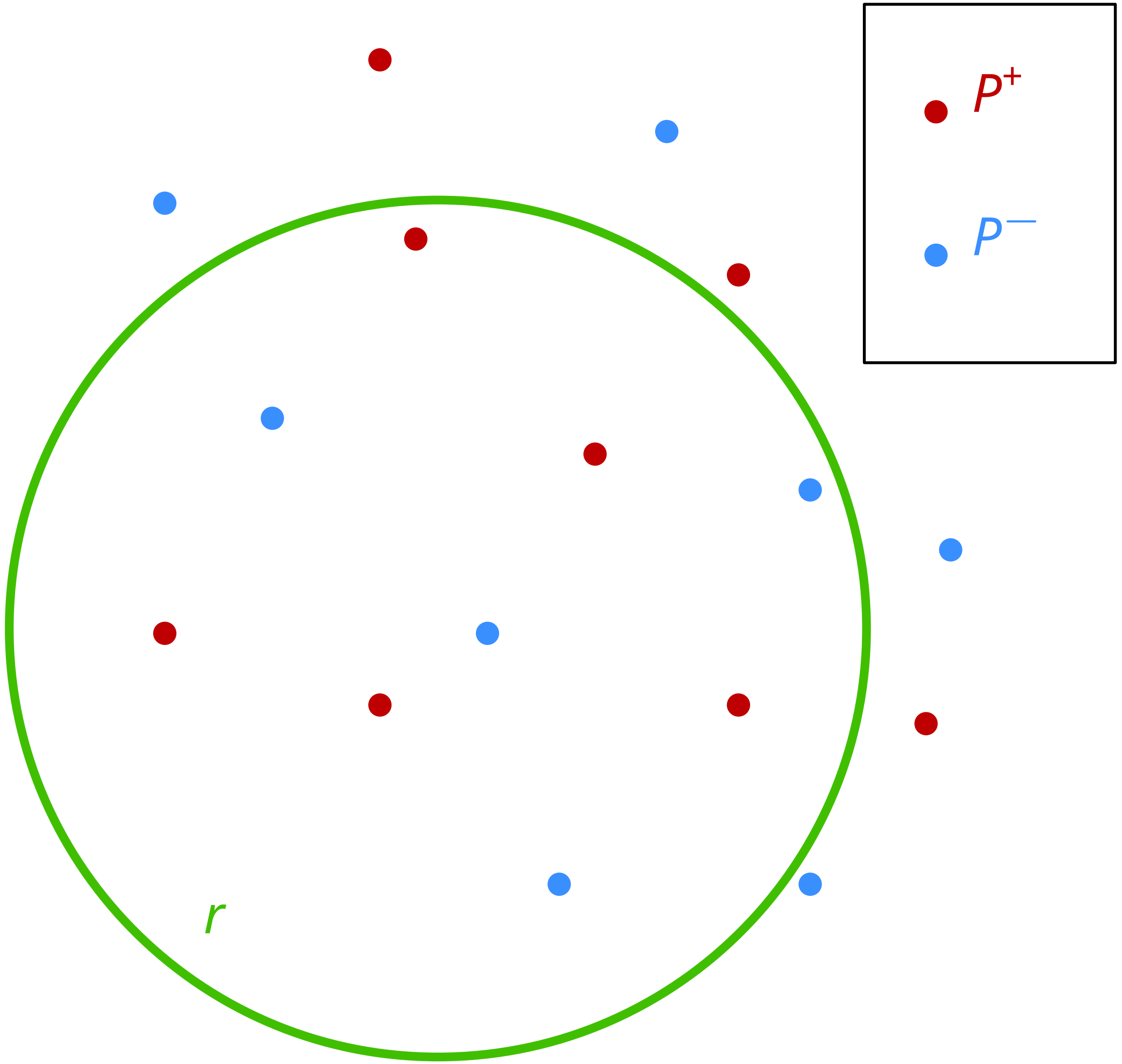

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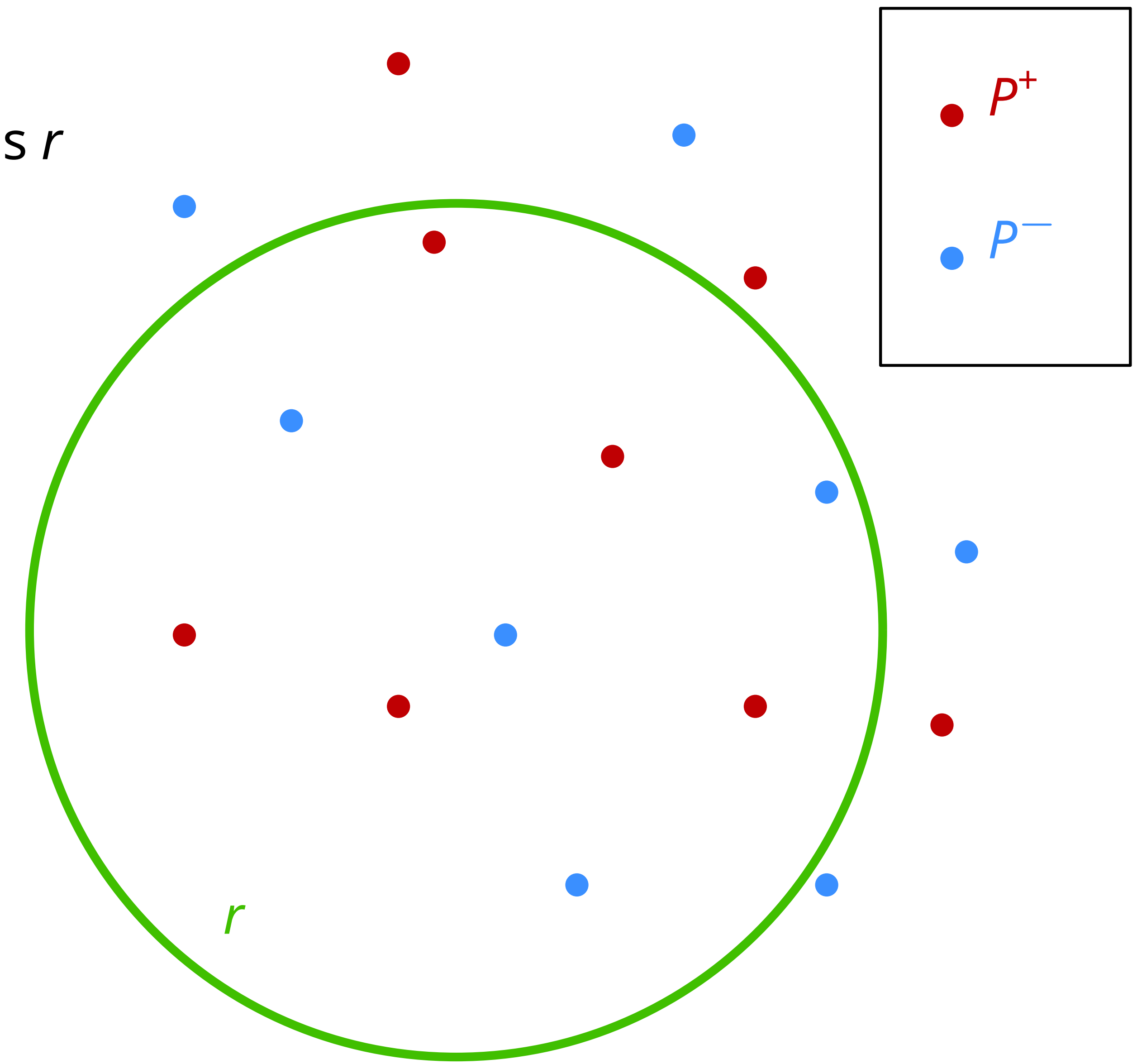
s.t. $|\chi(r)| = |\text{red} - \text{blue}|$ is small for all ranges r



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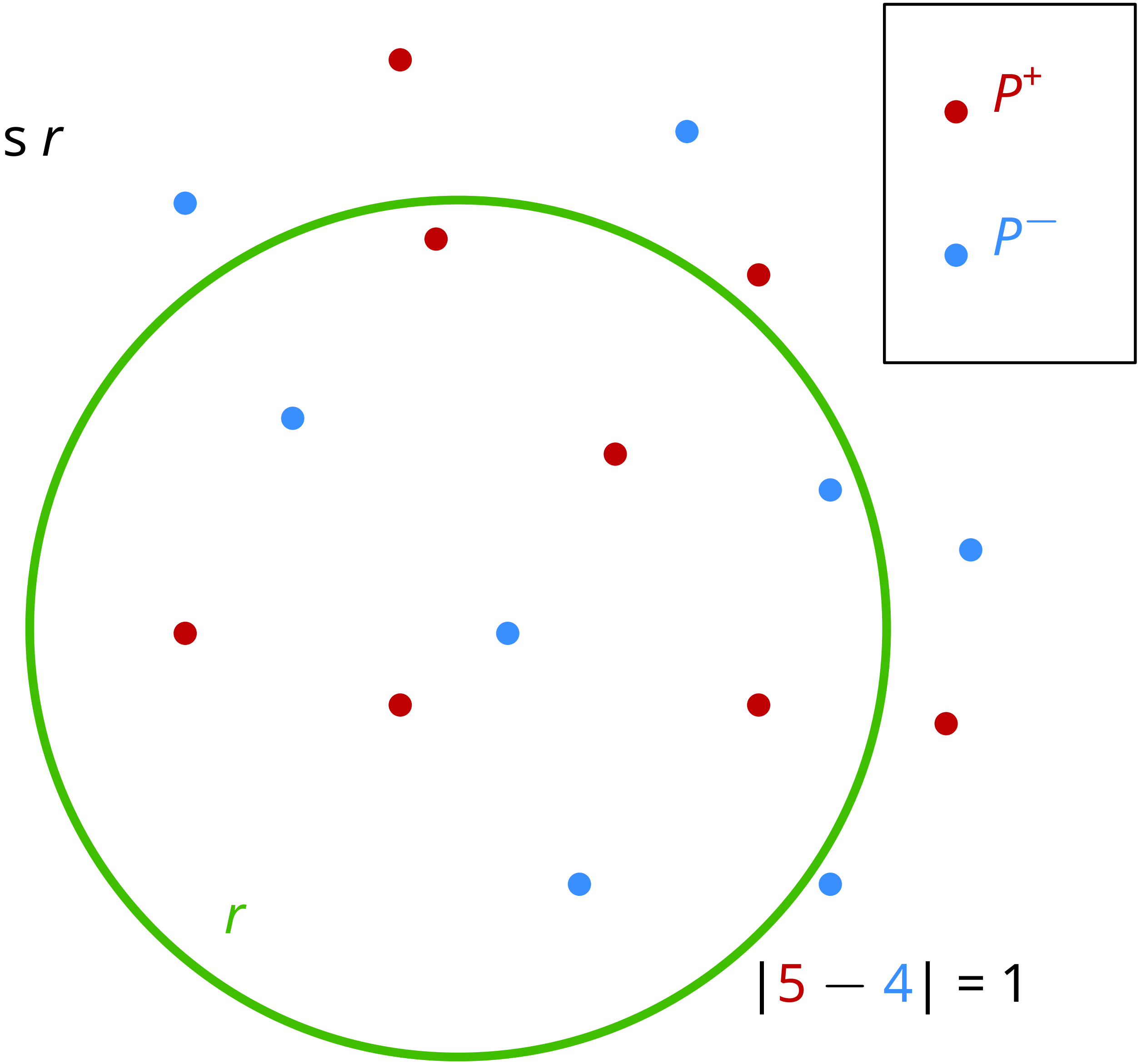

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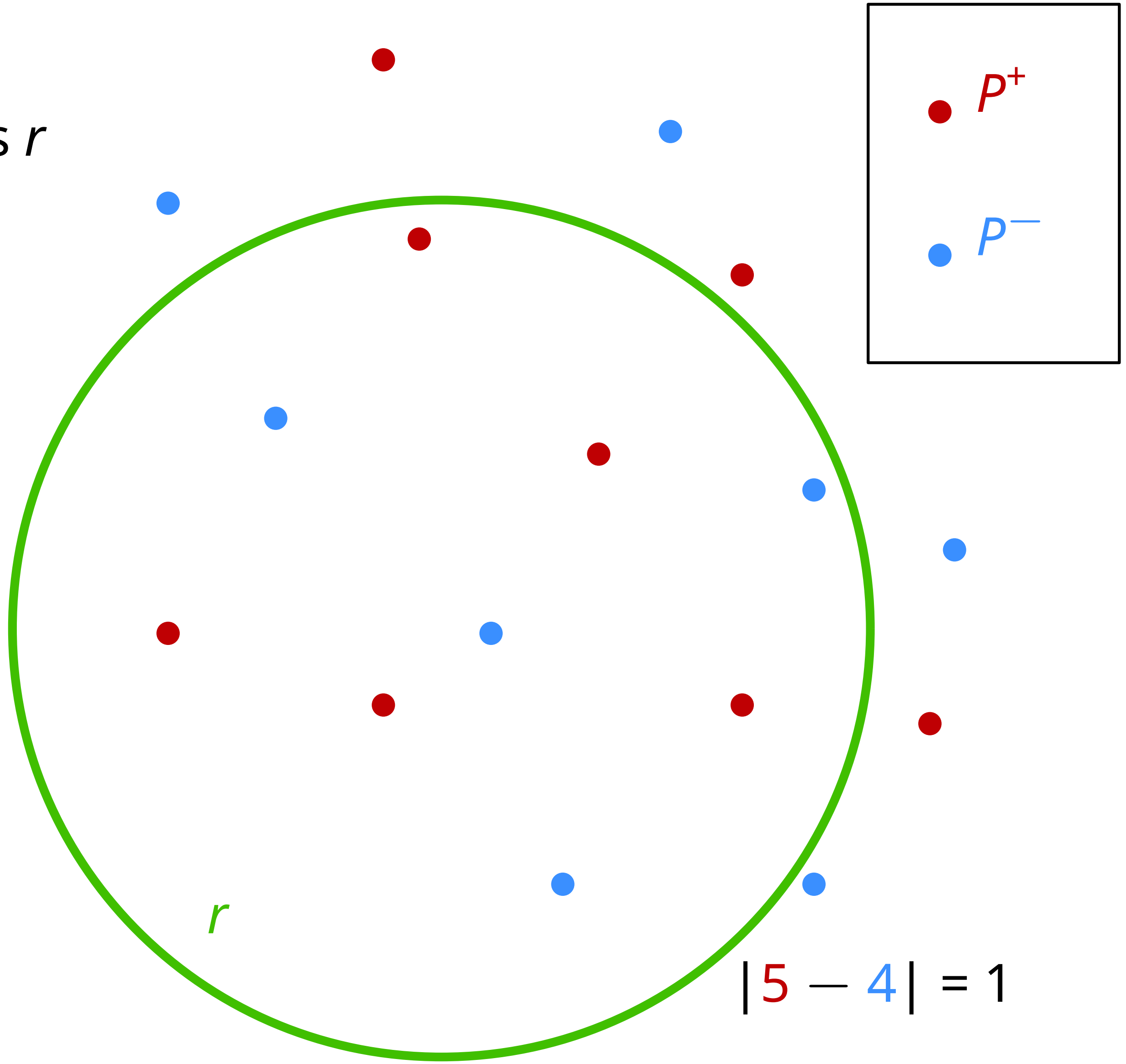
Quiz What is $\max_{r \in \mathcal{R}} |\chi(r)|$ in this example?

A 2

B 3

C 4



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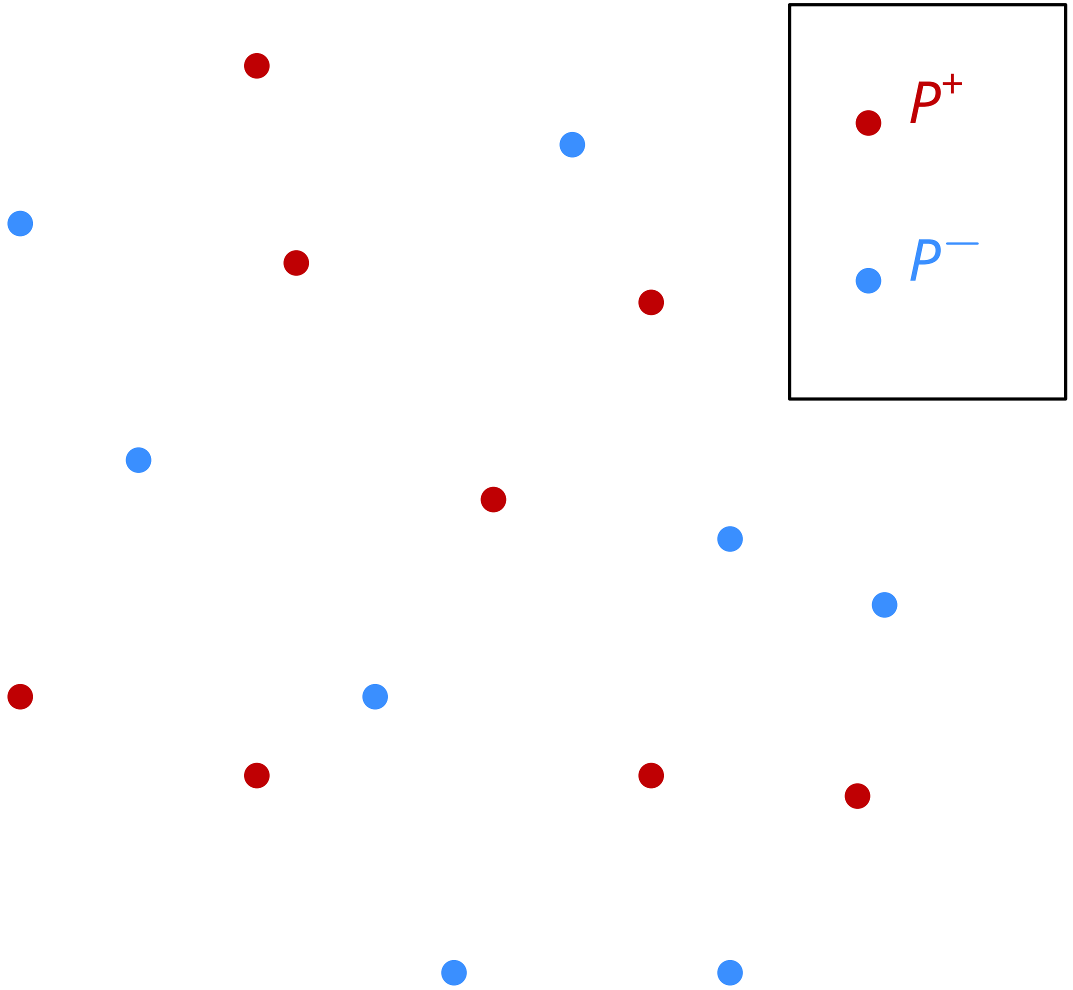

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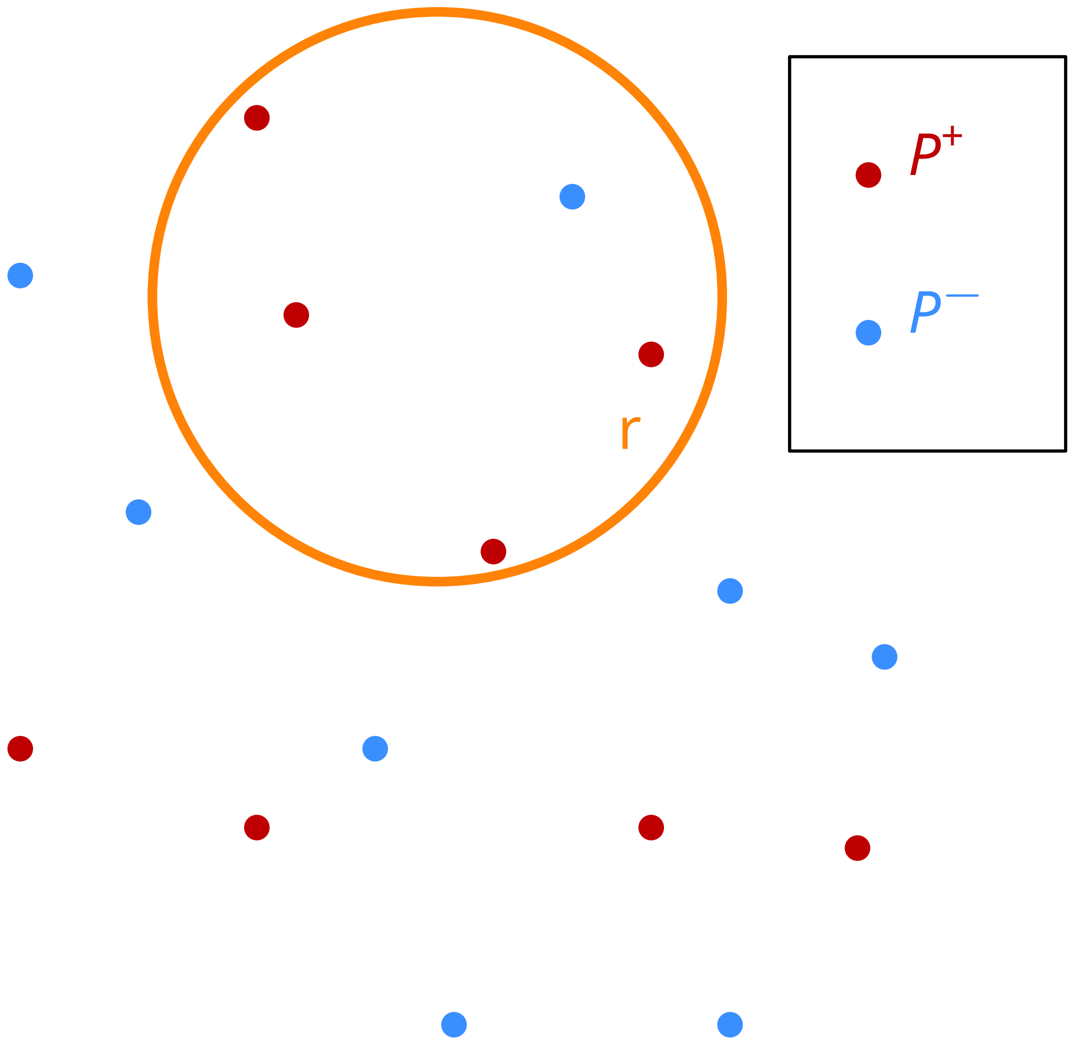

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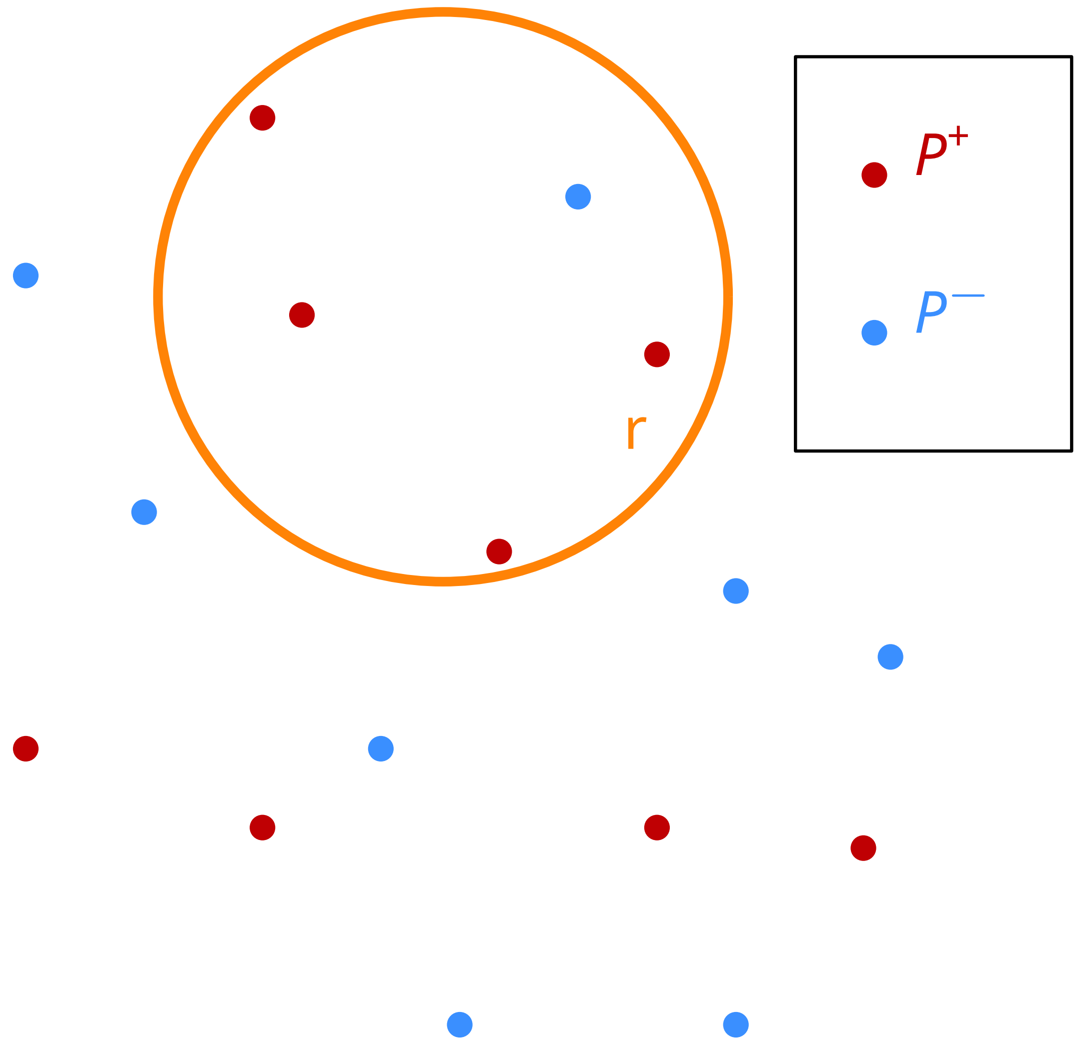

Formally:

coloring $\chi: X \to \{-1, 1\}$

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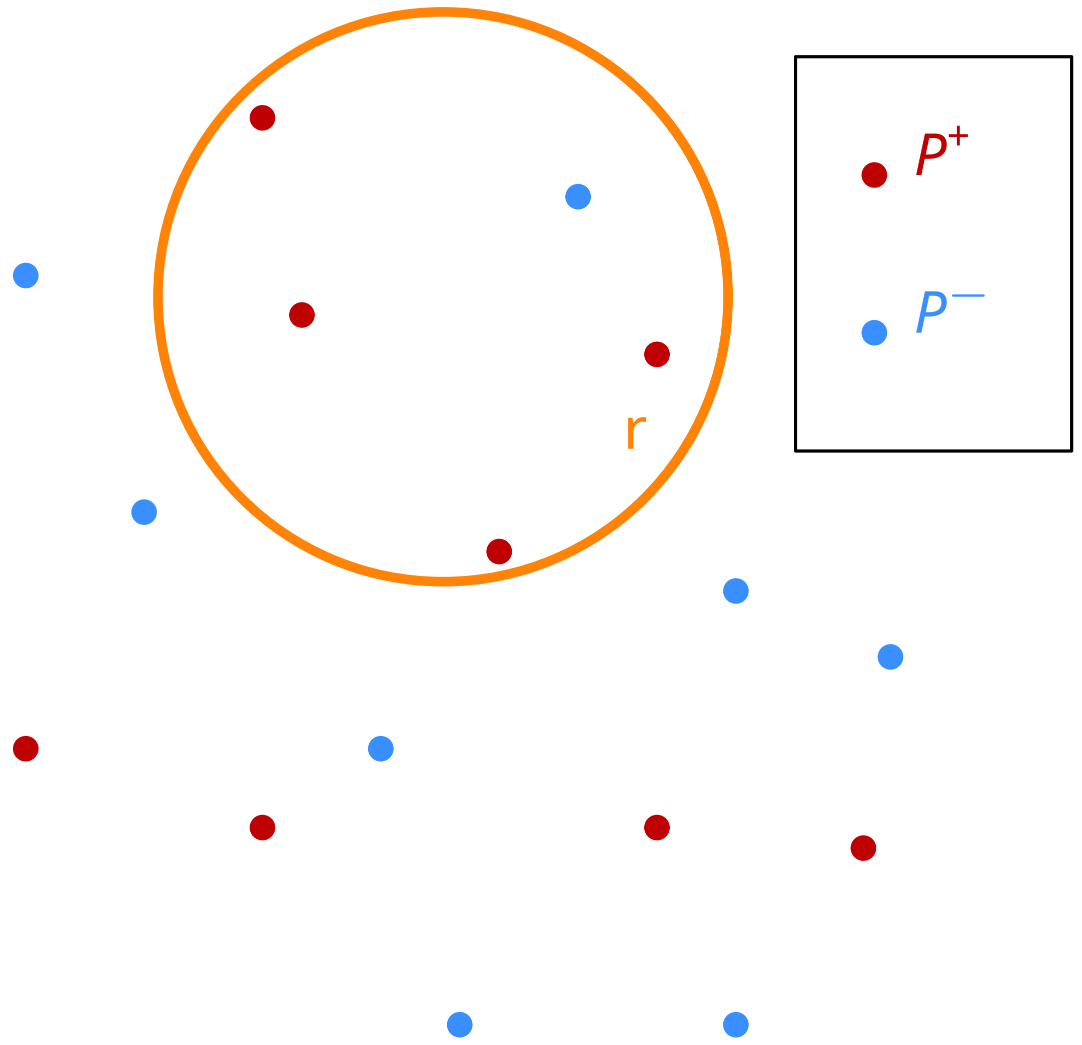

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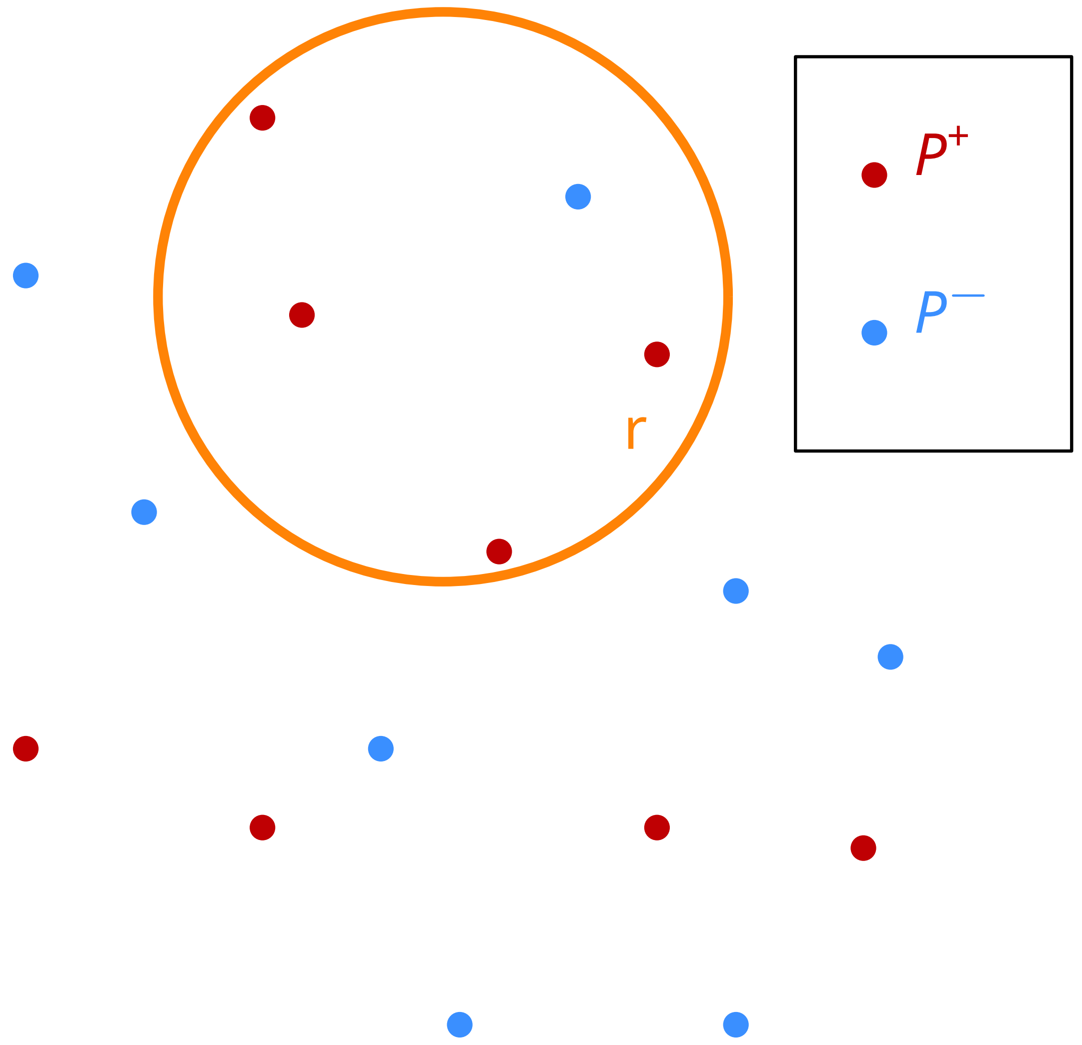

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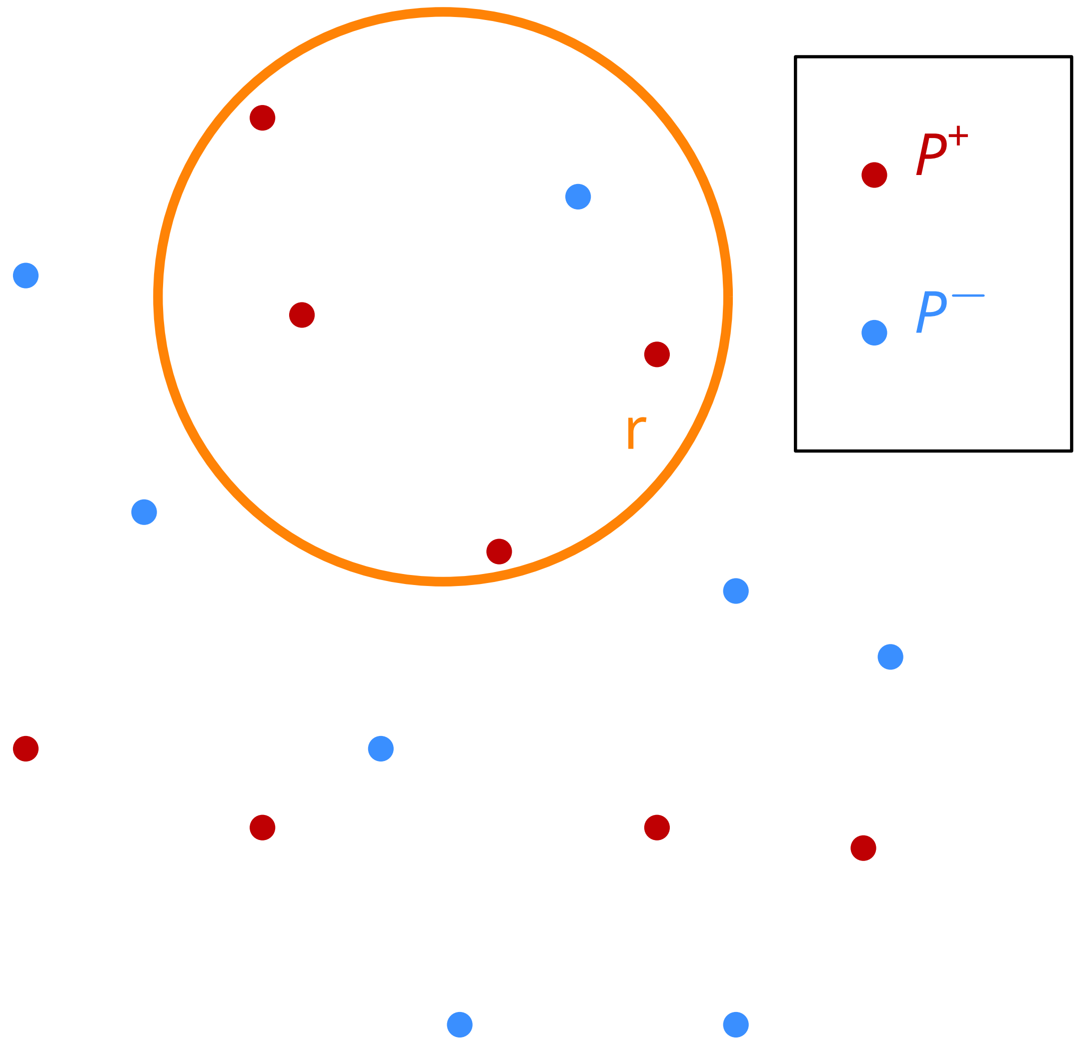
discrepancy of range space $S = (X, \mathcal{R})$:

$$\mathrm{disc}(S) = \min_{\chi: X \to \{-1,1\}} \mathrm{disc}(\chi)$$



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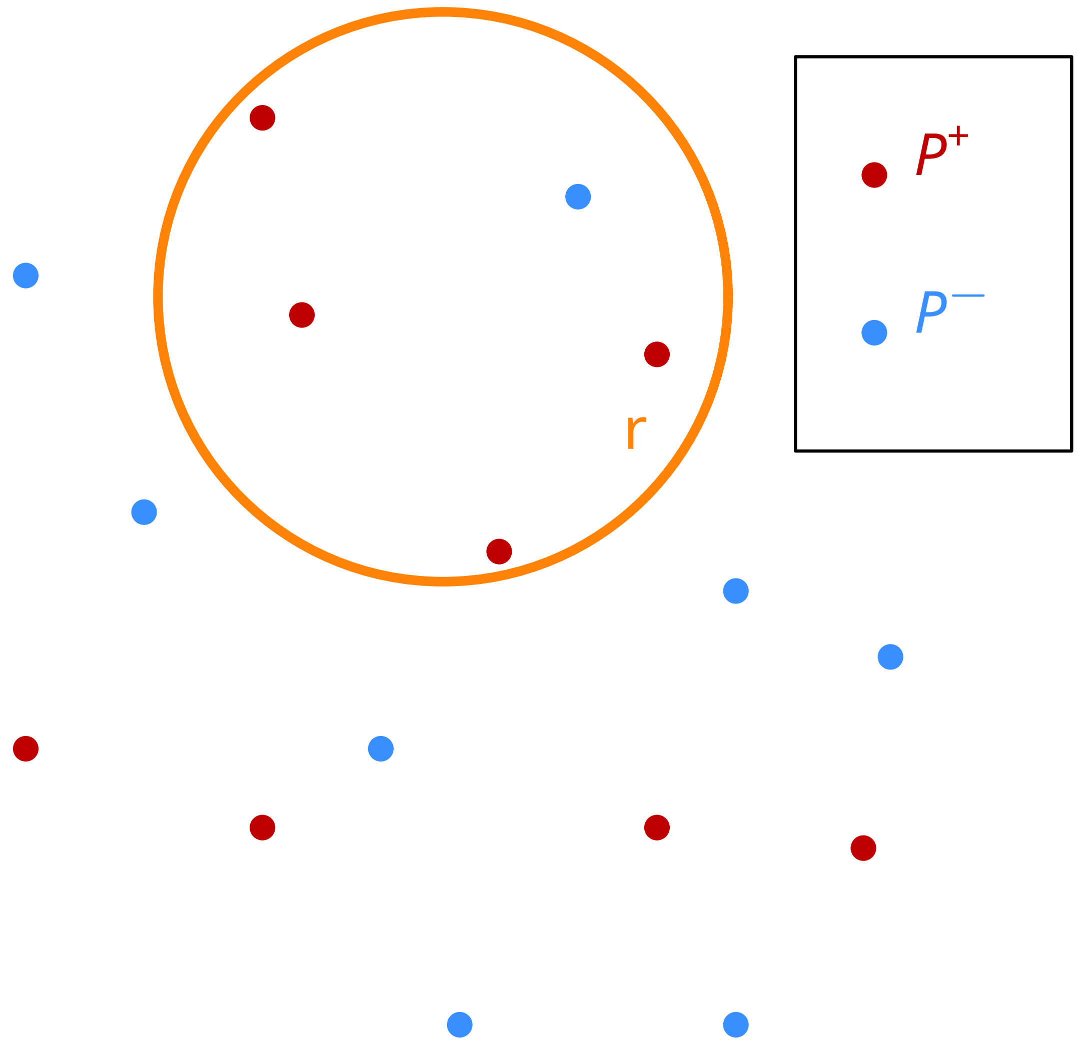

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Our goal: Given S, compute χ with low discrepancy

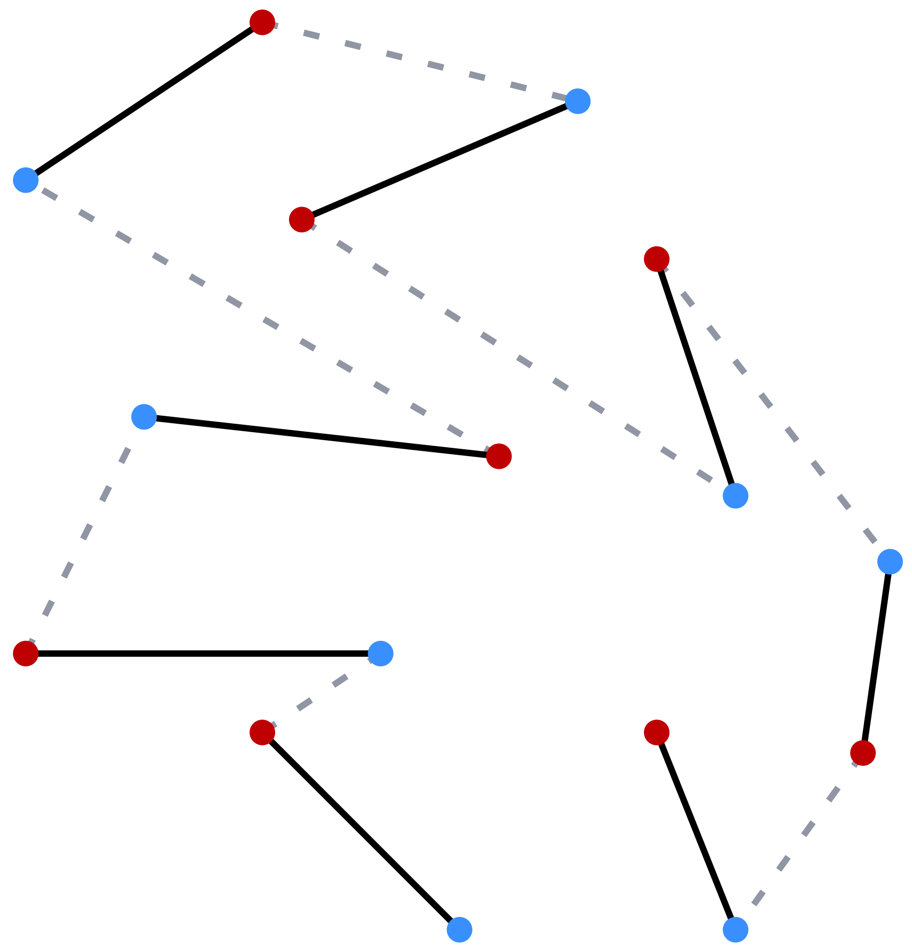
Our plan for today

From low discrepancy to ε-samples

Low-discrepancy colorings via perfect matchings & crossing numbers

Constructing a spanning tree with low crossing number

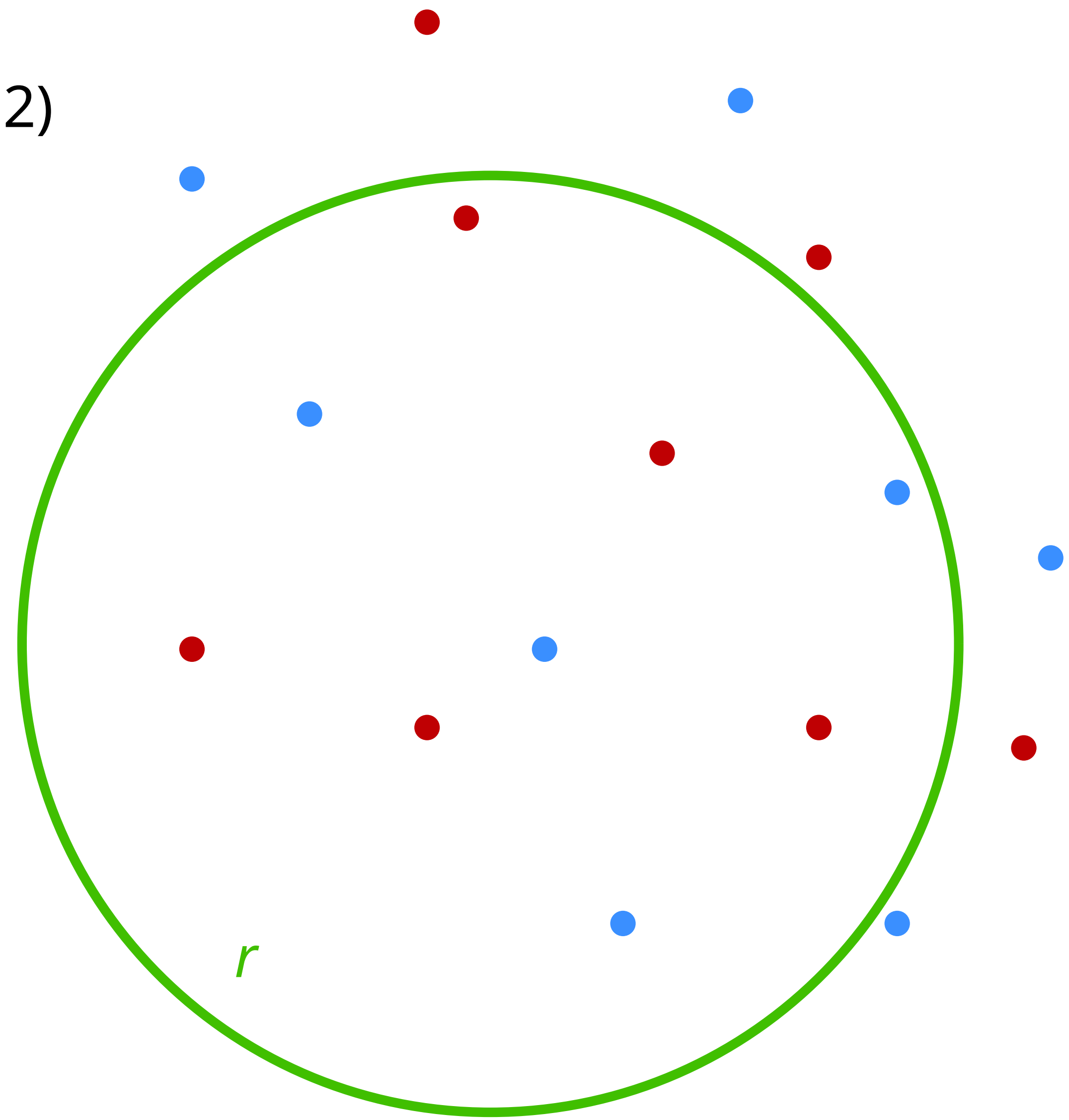
From spanning trees to perfect matchings



From low discrepancy to ε-samples

Assumptions:

$|P| = n$ is a power of 2. (correct up to factor 2)



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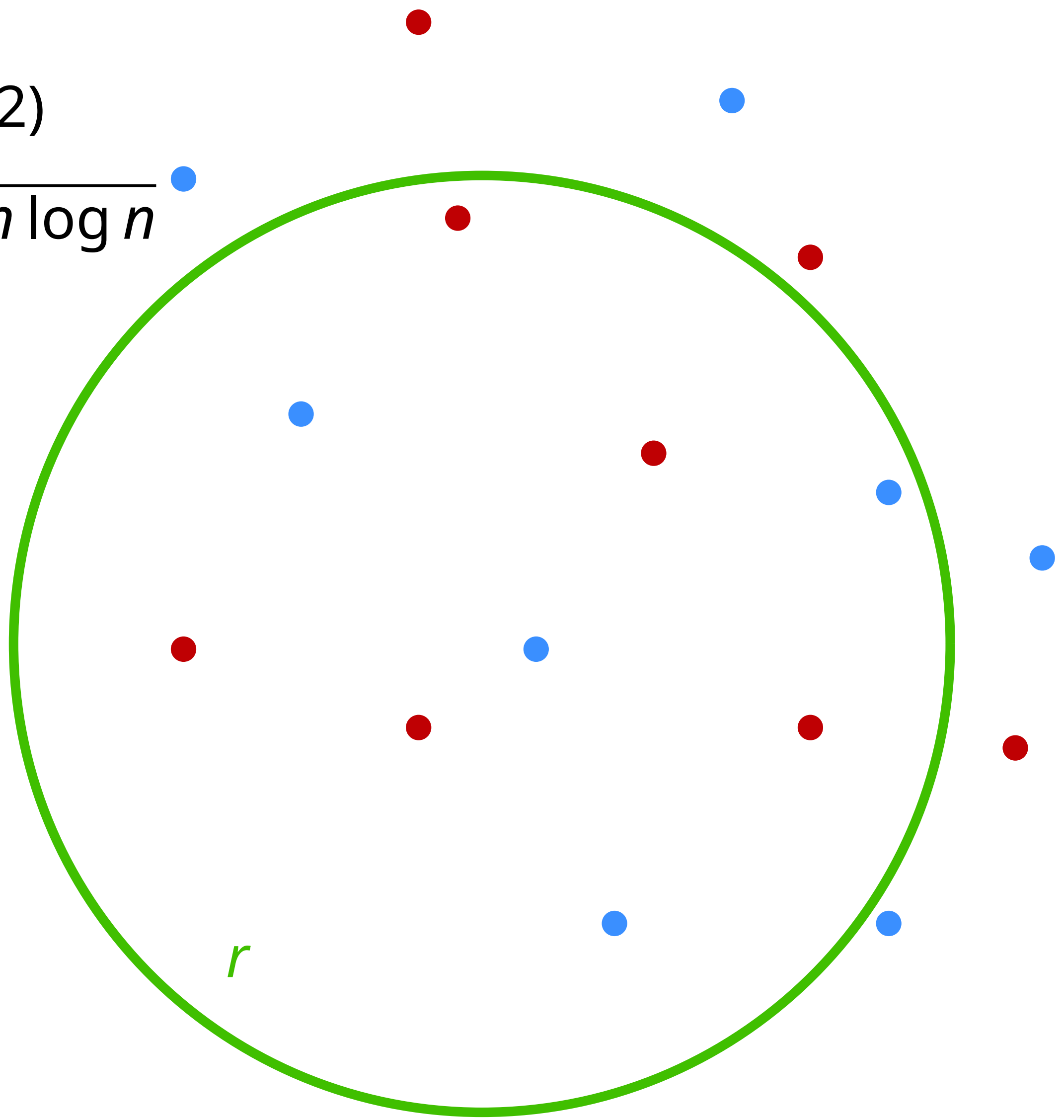
Assumptions:

$|P| = n$ is a power of 2. (correct up to factor 2)

We can find a coloring χ with $\text{disc}(\chi) \leq c\sqrt{\delta n \log n}$



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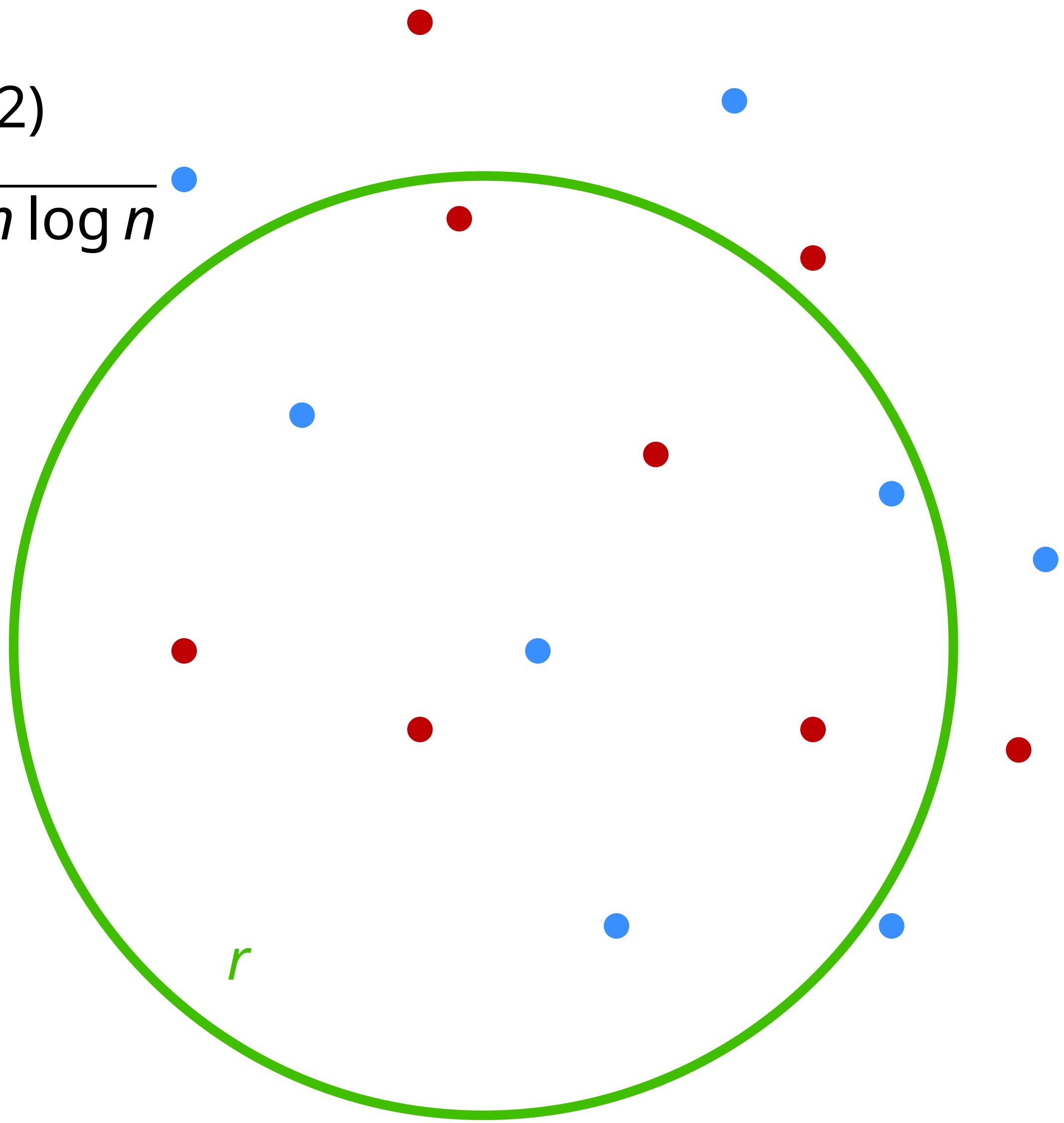
P_1: blue points ($\chi(p) = -1$), P'_1: red points



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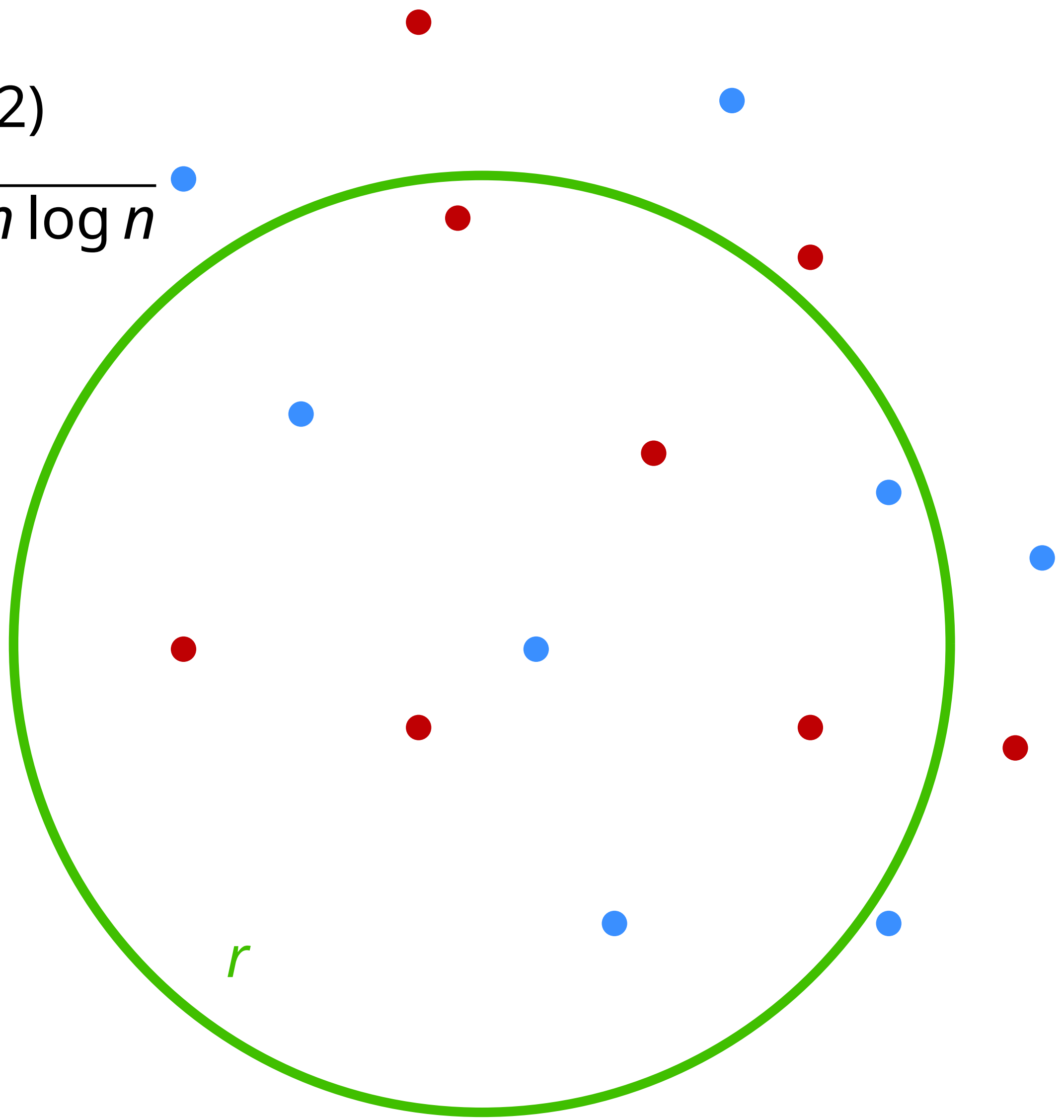
P_1 should be a good, but huge ε-sample



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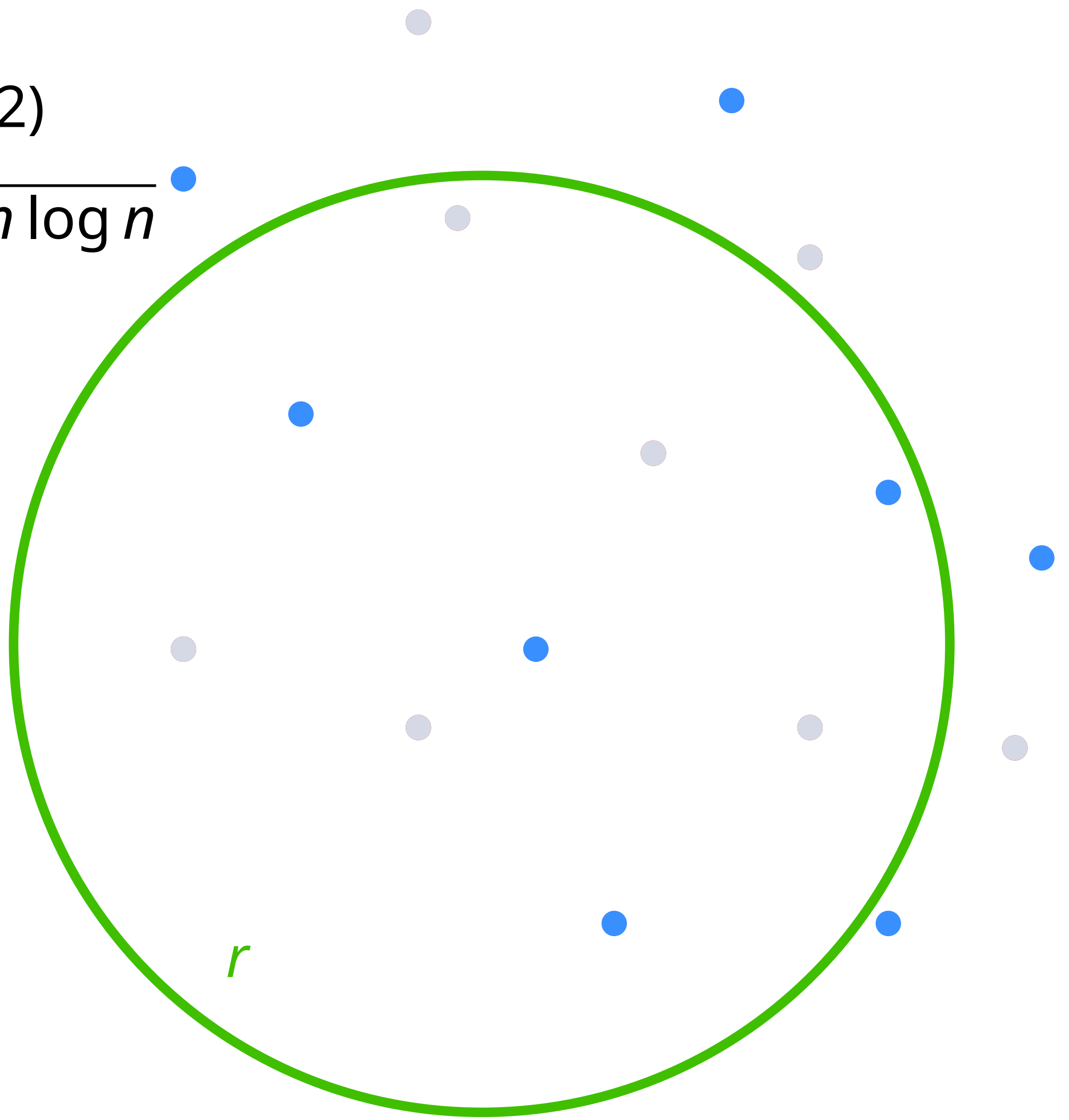
Iterate:

Compute χ_1 for P_1 with
$\text{disc}(\chi_1) \leq c\sqrt{\delta(n/2)\log(n/2)}$



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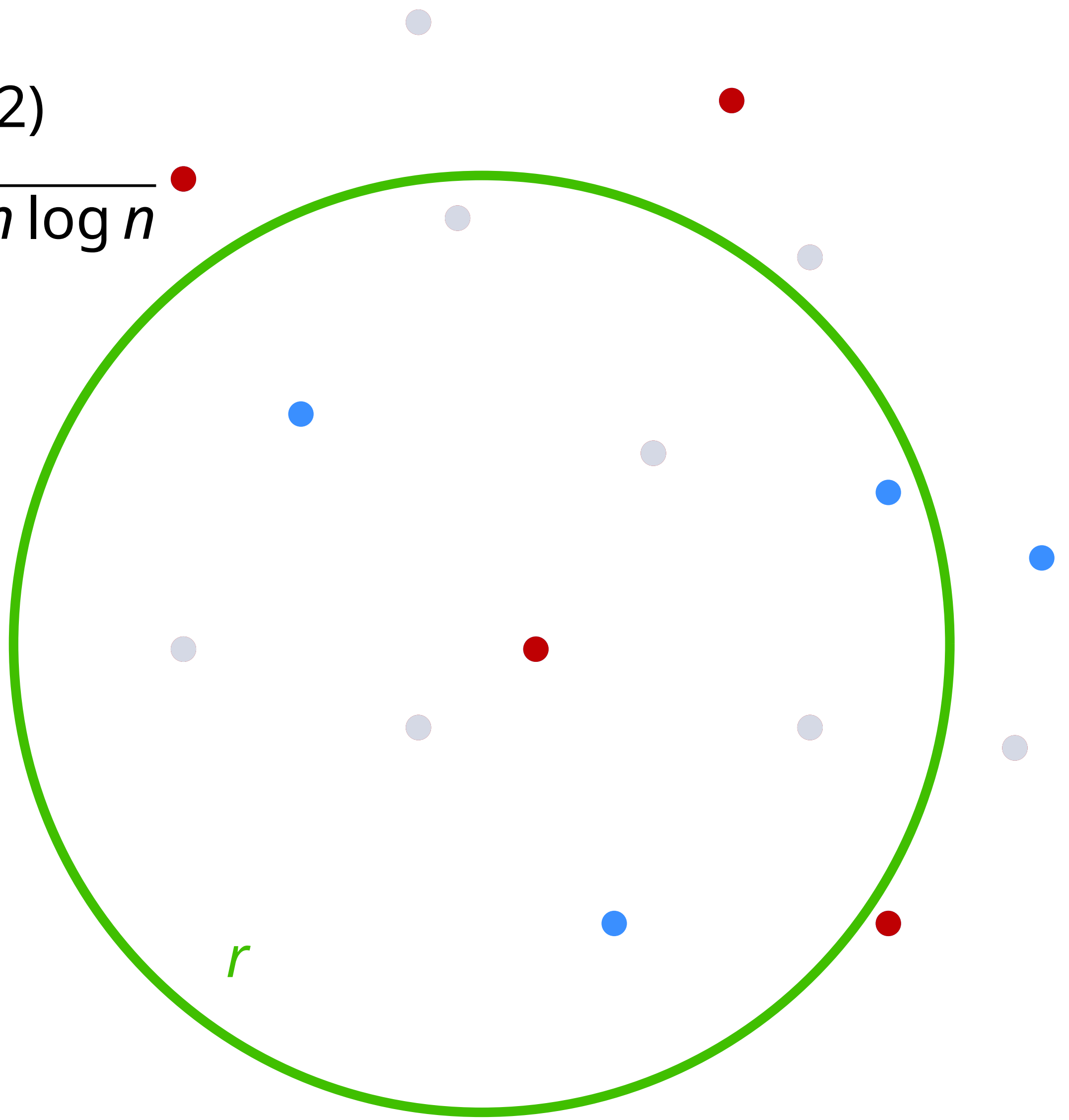
P_2: blue points ($\chi_1(p) = -1$)



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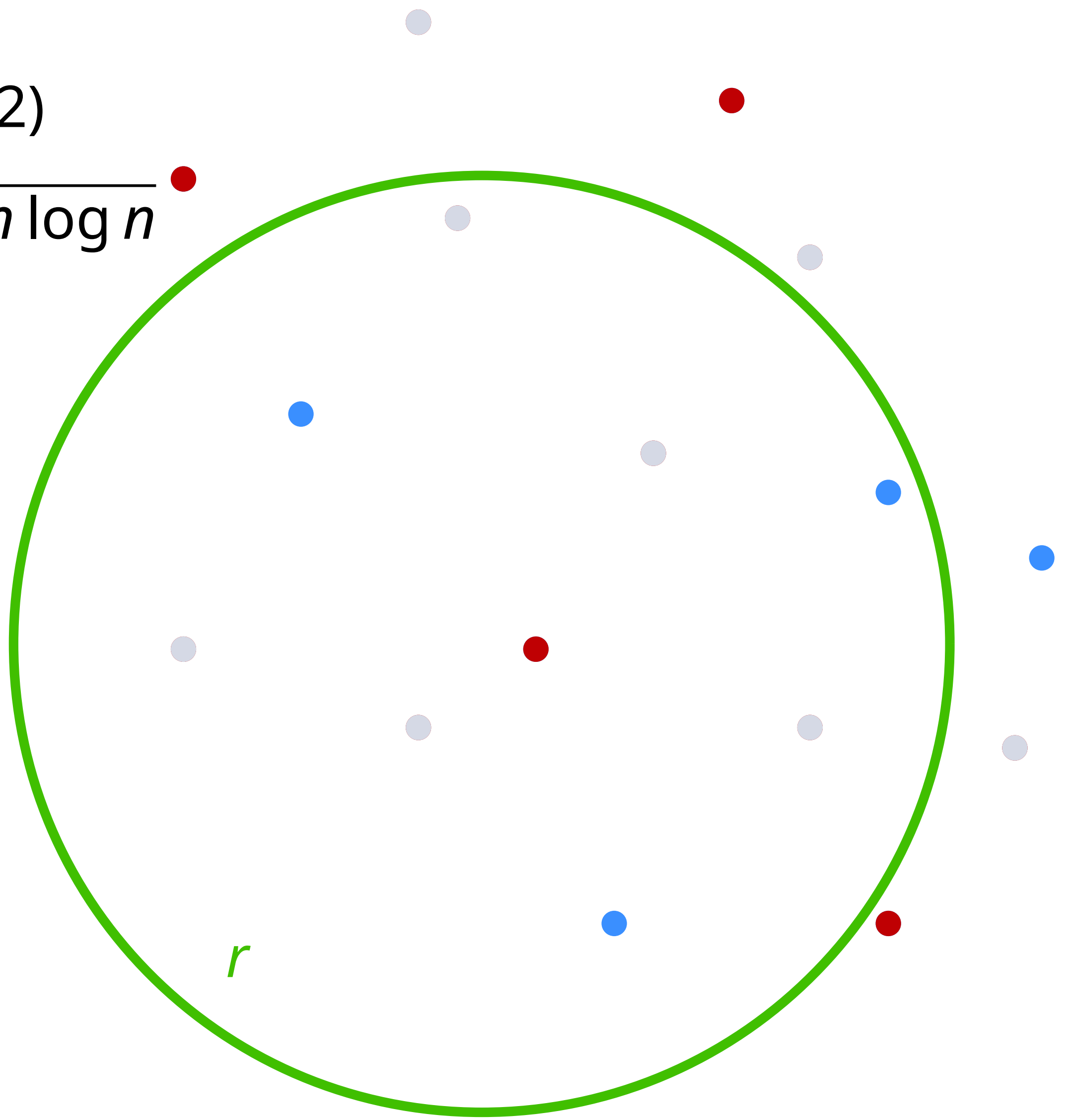
P_2 should be a good and smaller ε-sample



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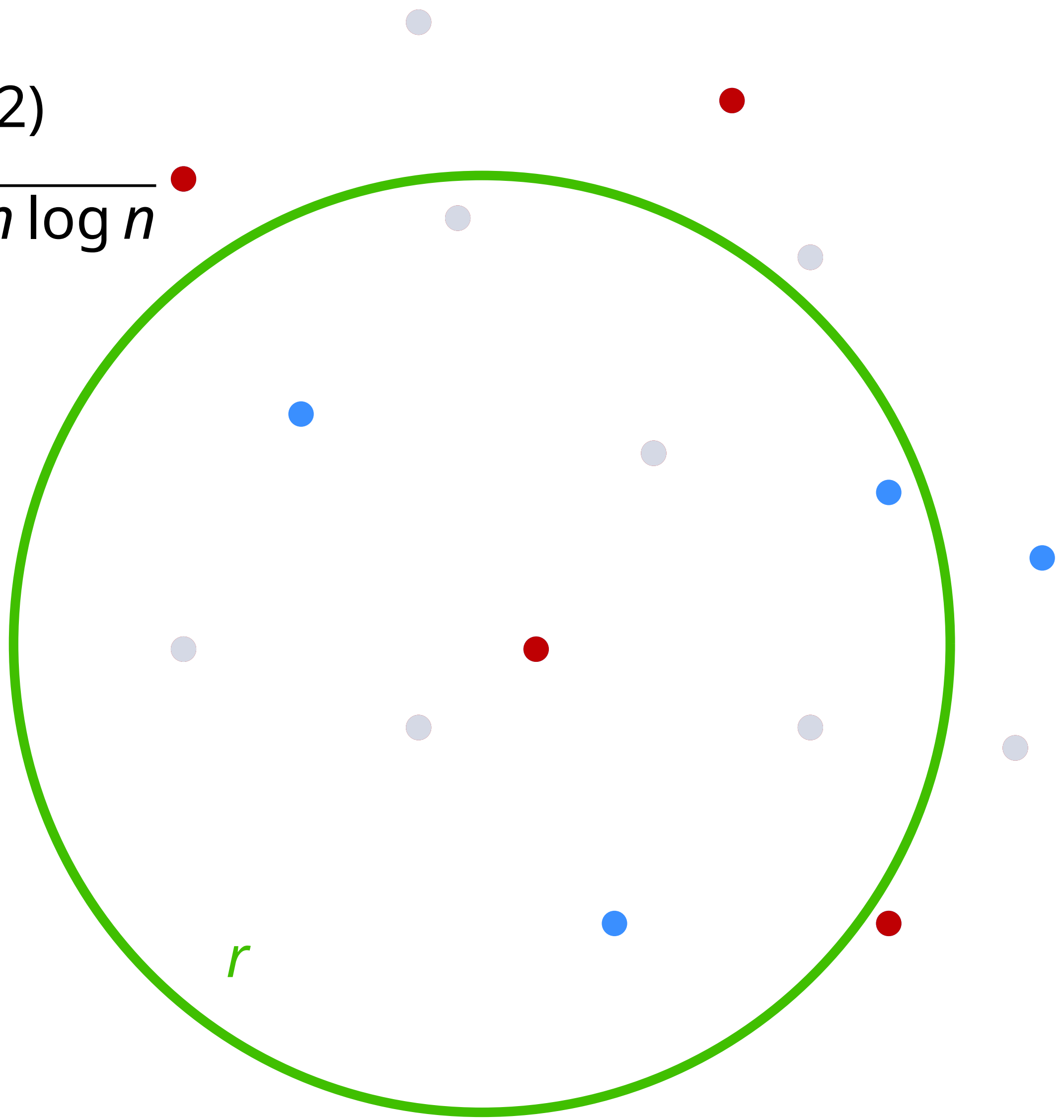
P_2 should be a good and smaller ε-sample

$P_3, P_4, \ldots$ How long can we iterate?



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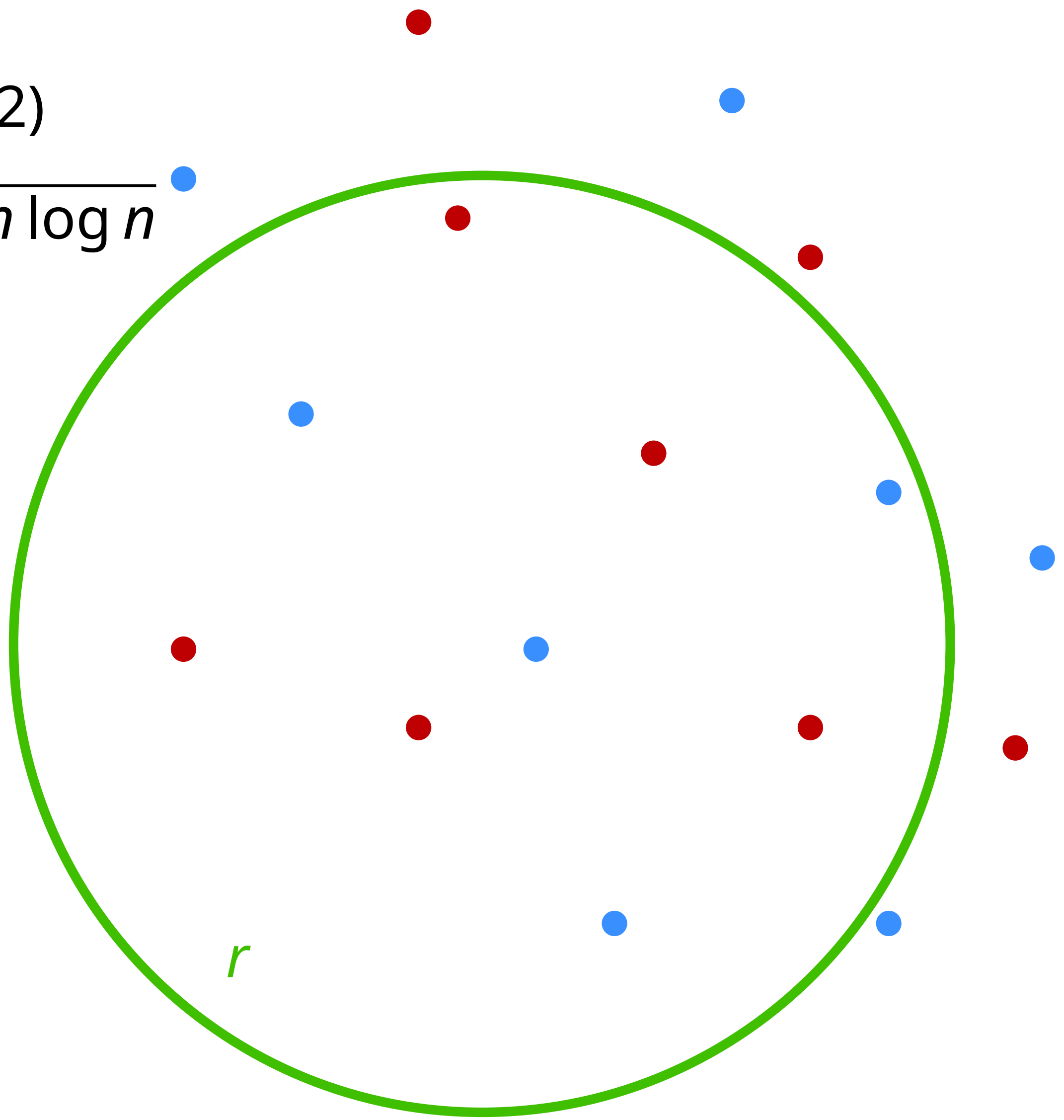

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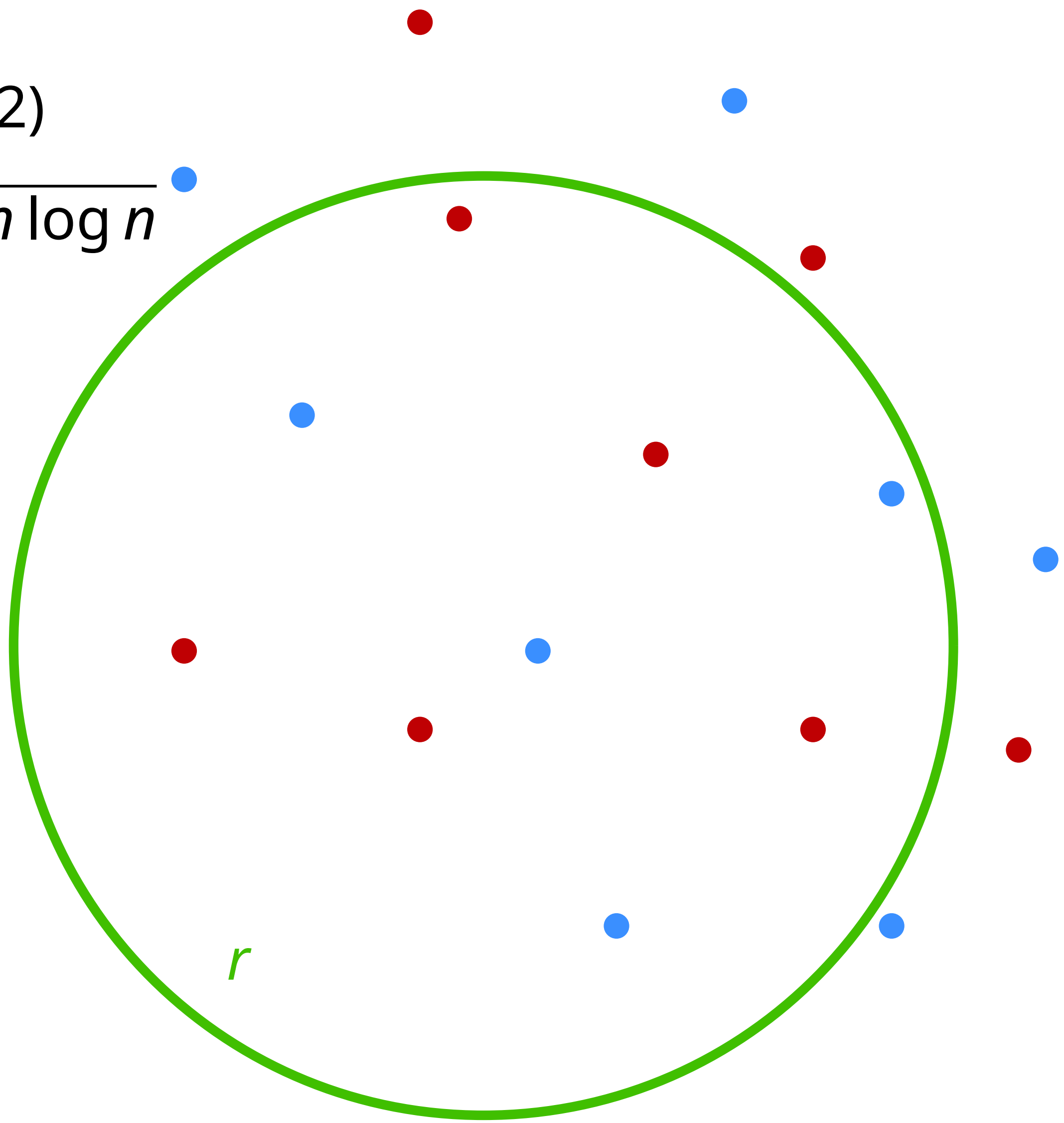
$$\left| \frac{|r \cap P|}{|P|} - \frac{|r \cap P_1|}{|P_1|} \right| =$$



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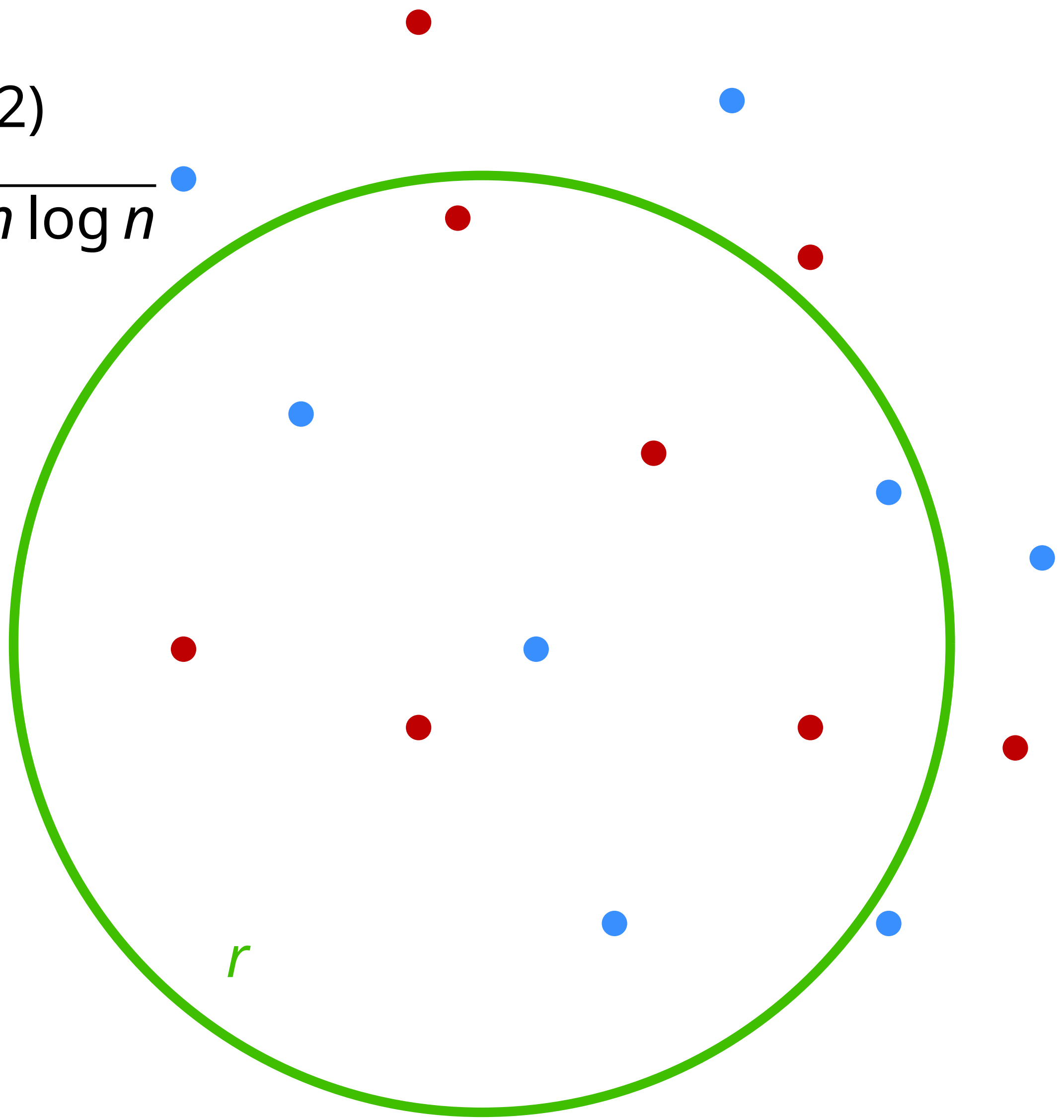
$$\left| \frac{|r \cap P|}{|P|} - \frac{|r \cap P_1|}{|P_1|} \right| = \frac{1}{n} \left| |r \cap P| - 2|r \cap P_1| \right| =$$



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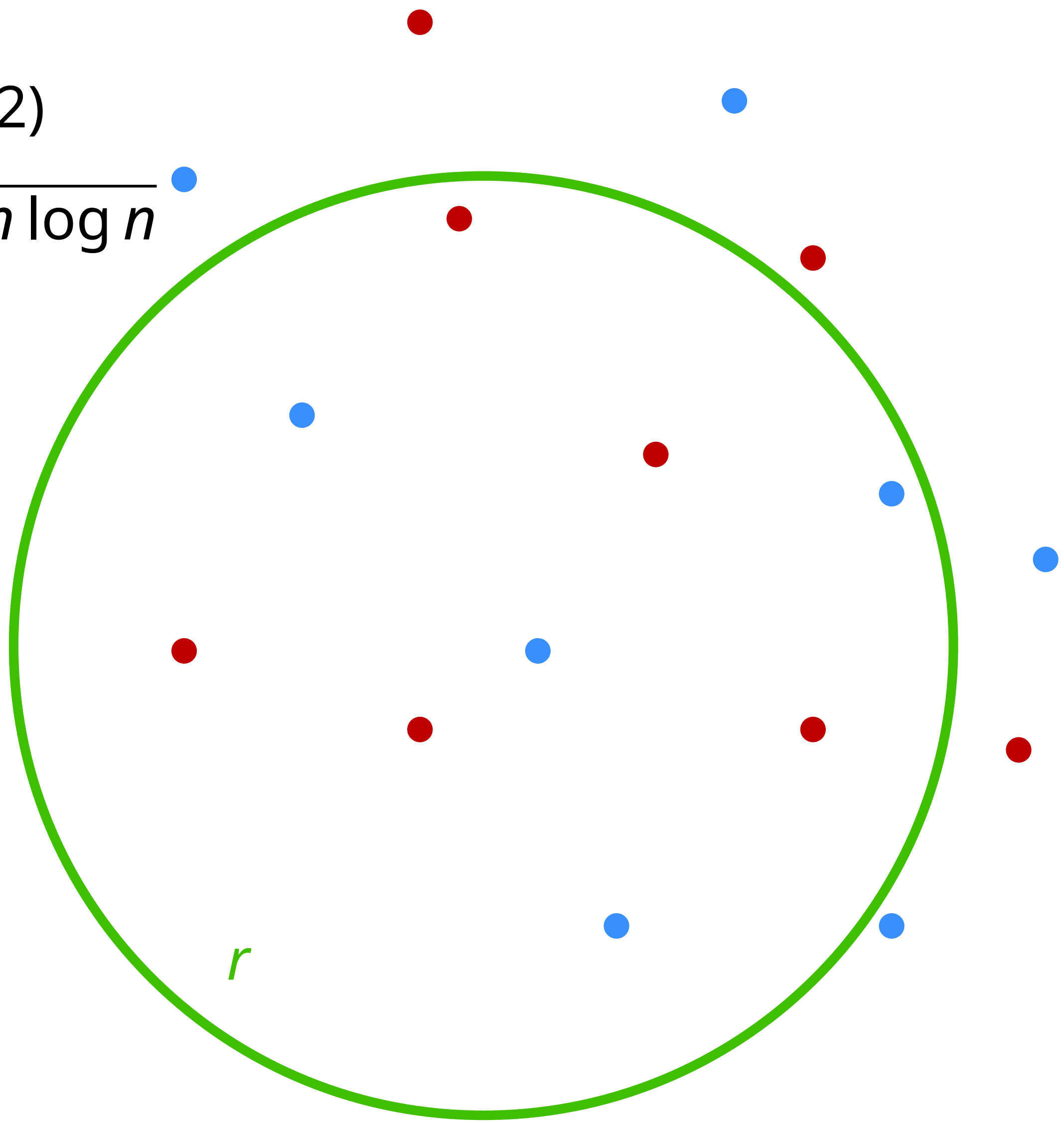
$$\frac{1}{n} \left| |r \cap P_2| + |r \cap P_1| - 2|r \cap P_1| \right| \leq$$



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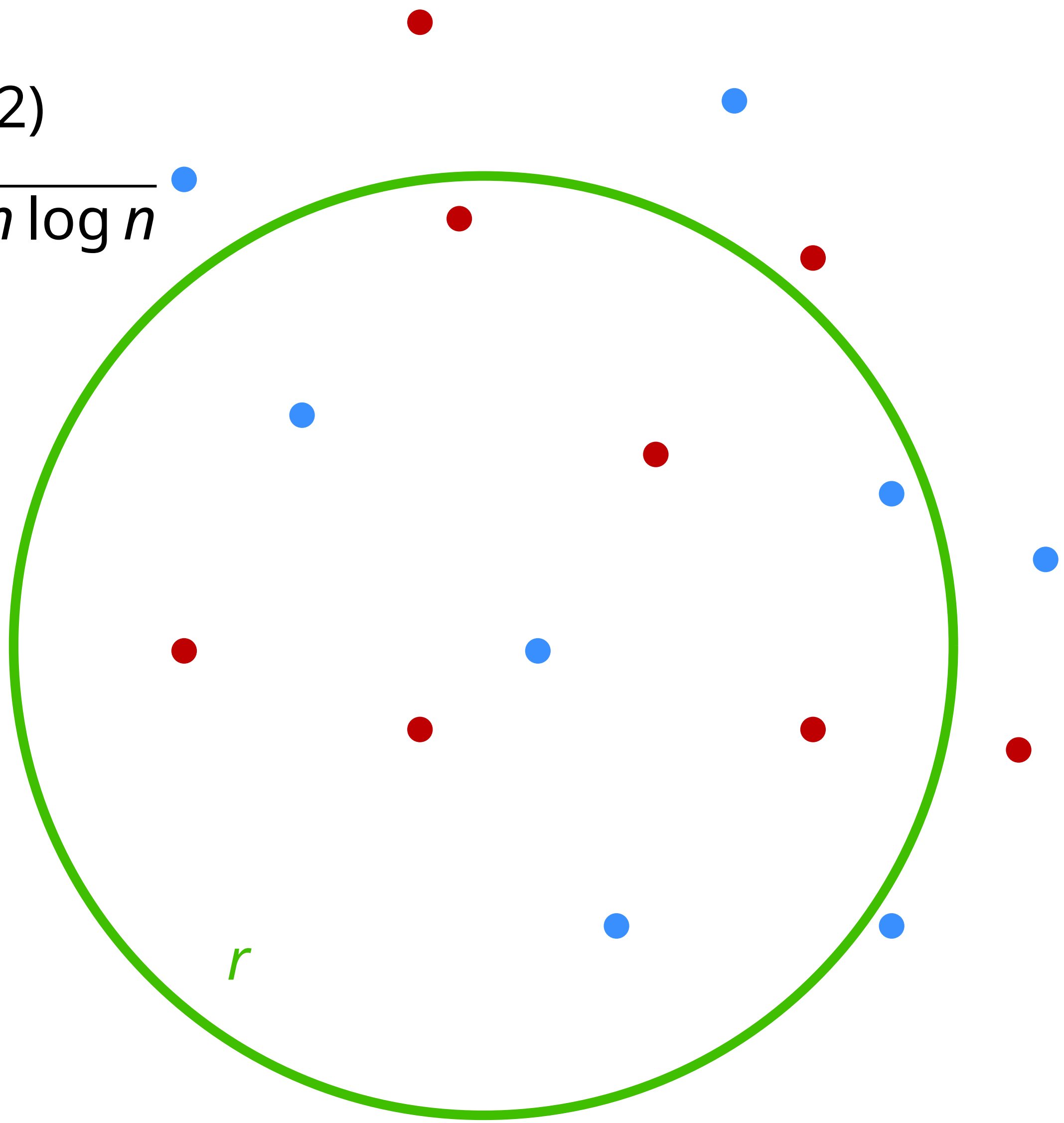
$$\frac{c\sqrt{\delta n \log n}}{n} = c\sqrt{\frac{\delta \log n}{n}}$$



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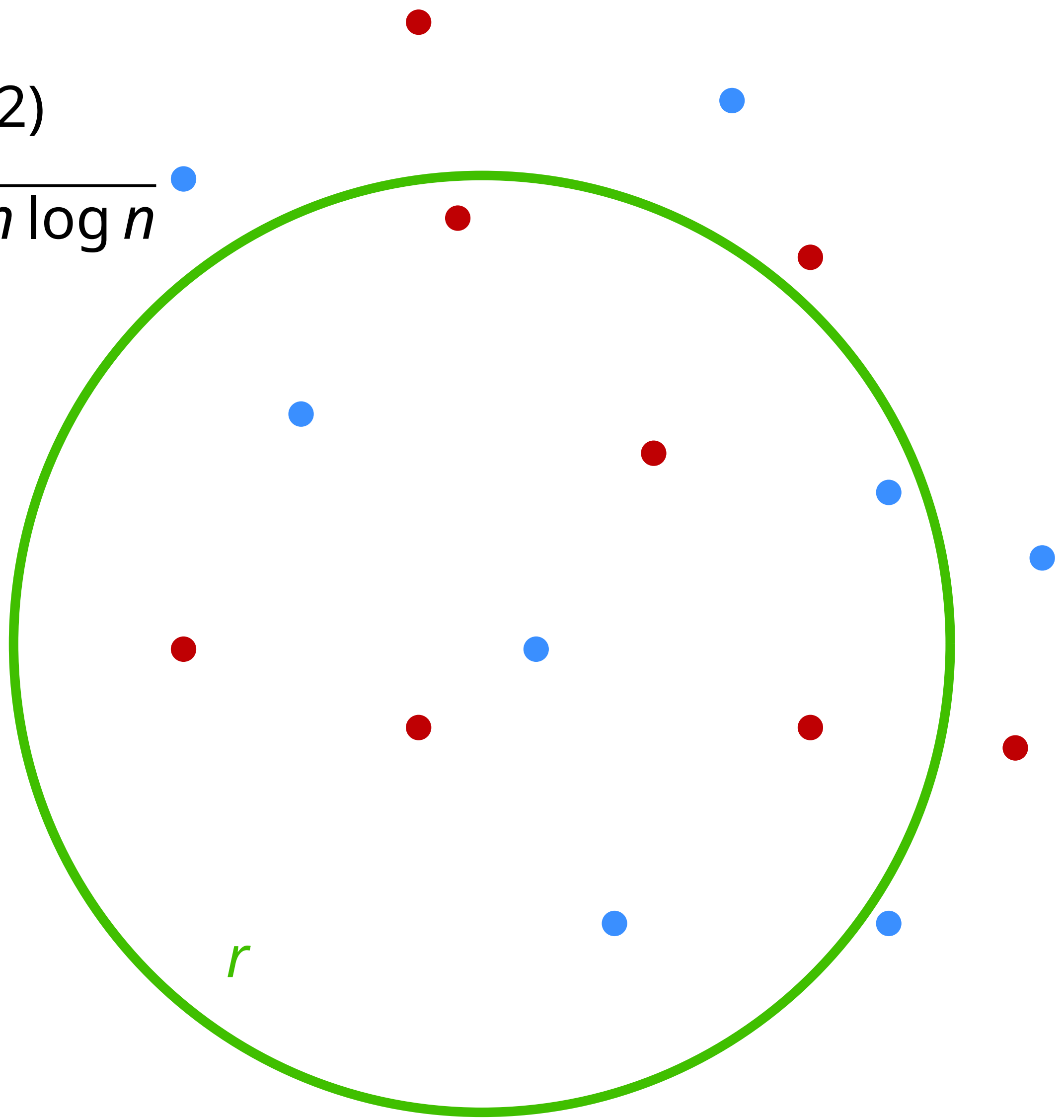

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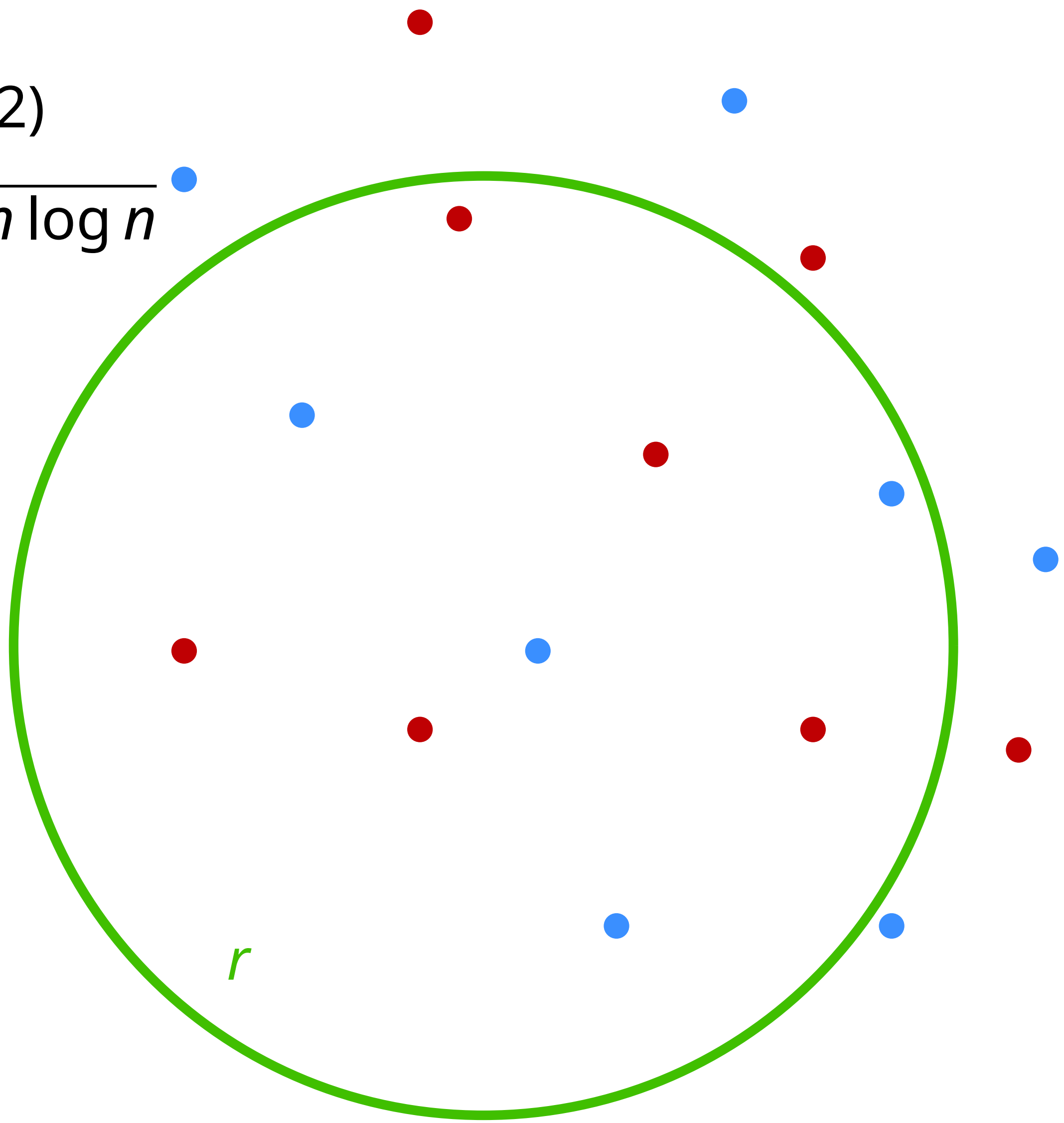
P_2 is ε'-sample of P_1 with $\varepsilon' = c\sqrt{\delta \frac{\log(n/2)}{n/2}}$



Quiz

If P_1 is an ε_1-sample of P, and P_2 is an ε_2-sample of P_1, then P_2 is an ...-sample of P.

A $\varepsilon_1 + \varepsilon_2$

B $\varepsilon_1 \cdot \varepsilon_2$

C $\max(\varepsilon_1, \varepsilon_2)$

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$$\left| \frac{|r \cap P|}{|P|} - \frac{|r \cap P_2|}{|P_2|} \right| = \left| \frac{|r \cap P|}{|P|} - \frac{|r \cap P_1|}{|P_1|} + \frac{|r \cap P_1|}{|P_1|} - \frac{|r \cap P_2|}{|P_2|} \right| \leq$$

Quiz

If P_1 is an ε_1-sample of P, and P_2 is an ε_2-sample of P_1, then P_2 is an ...-sample of P.

A $\varepsilon_1 + \varepsilon_2$

B $\varepsilon_1 \cdot \varepsilon_2$

C $\max(\varepsilon_1, \varepsilon_2)$

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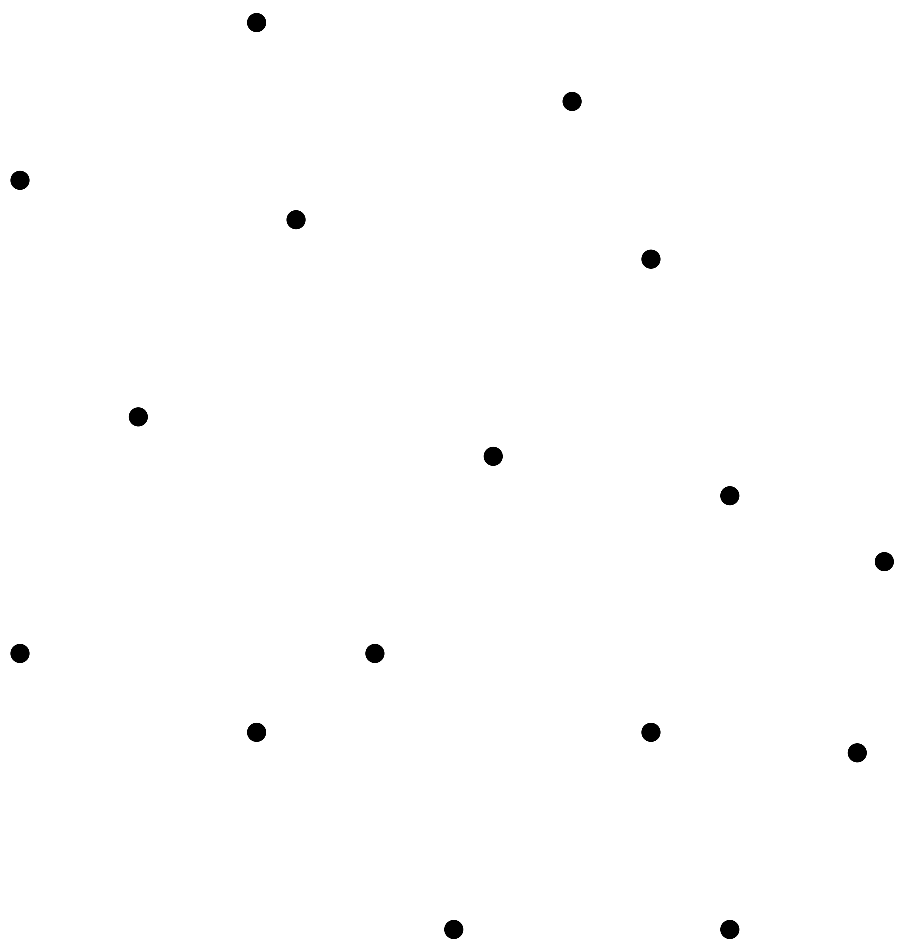
Gives ε-sample of size $O(\frac{\delta}{\varepsilon^2} \log \frac{\delta}{\varepsilon^2})$ if assumption holds: We can find a coloring χ with $\mathrm{disc}(\chi) \leq c\sqrt{\delta n \log n}$

Low-discrepancy colorings via perfect matchings & crossing numbers

Construction via perfect matchings

Assumption: $|P| = n$ is even



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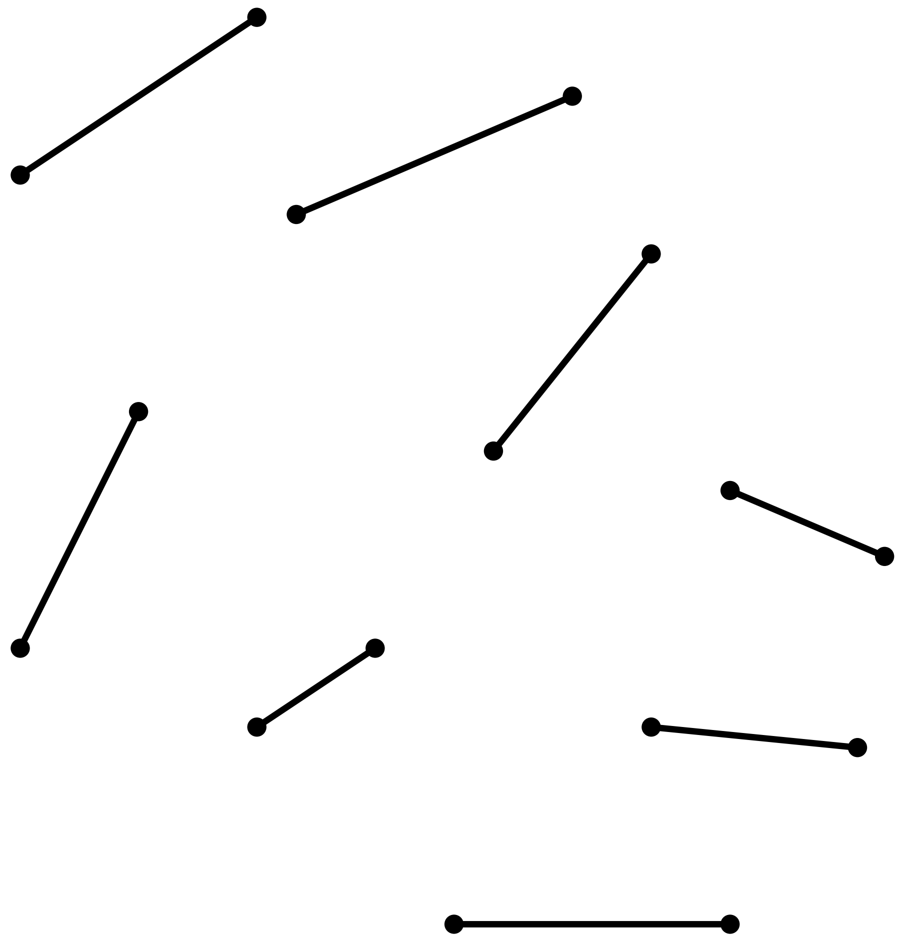

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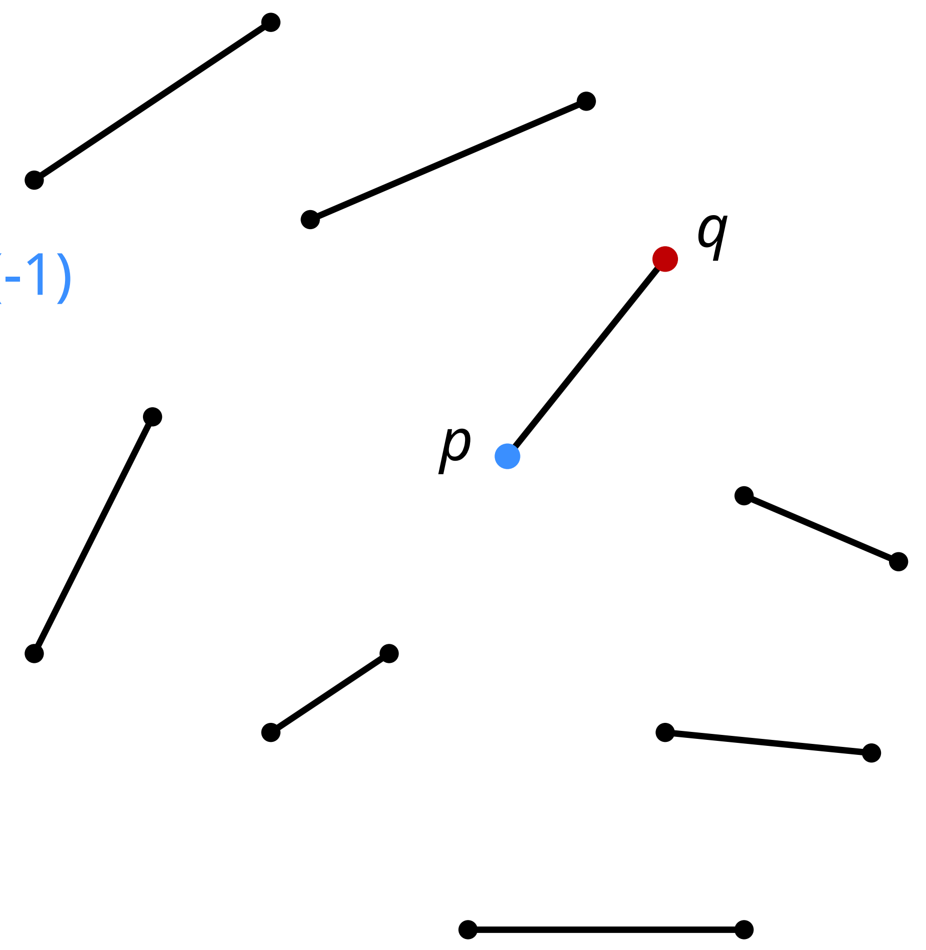

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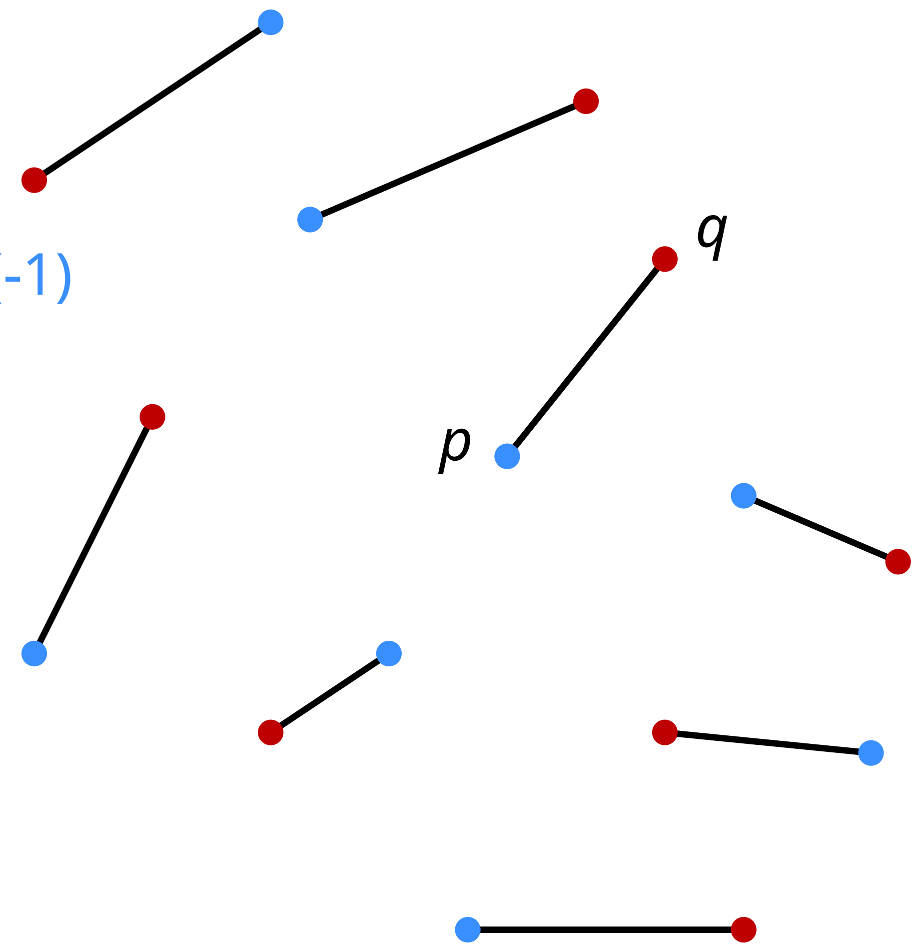

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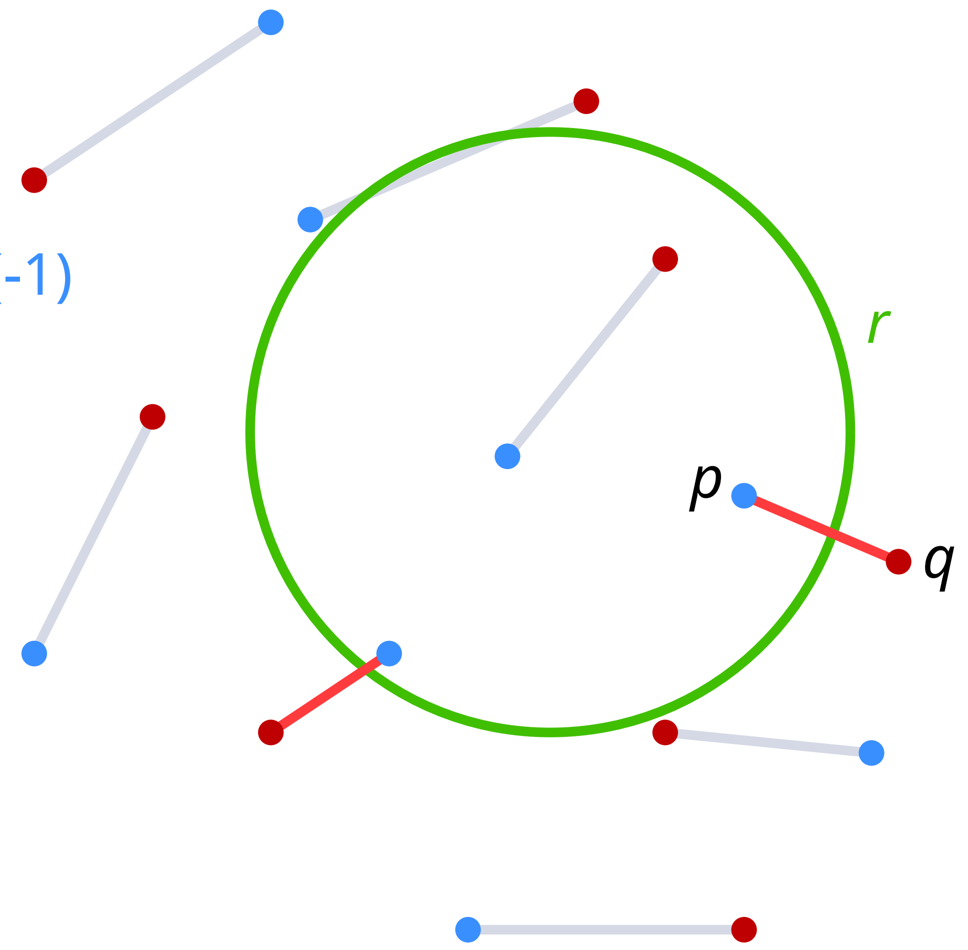

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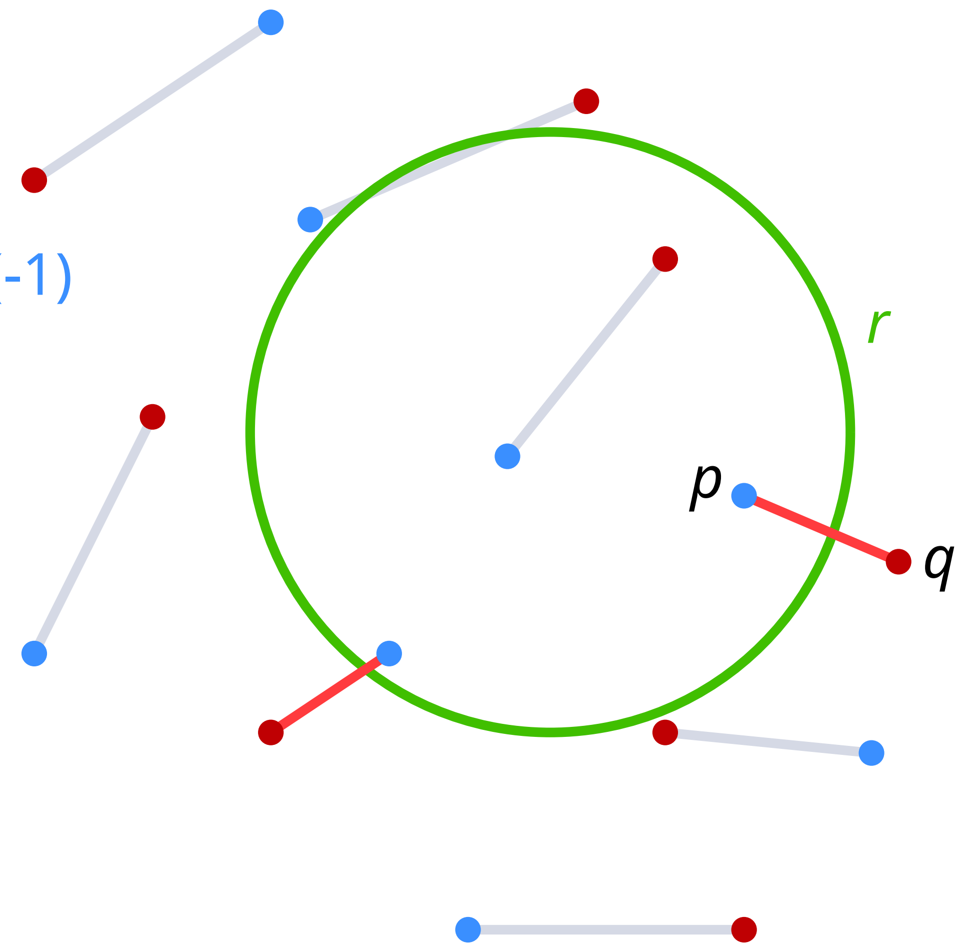

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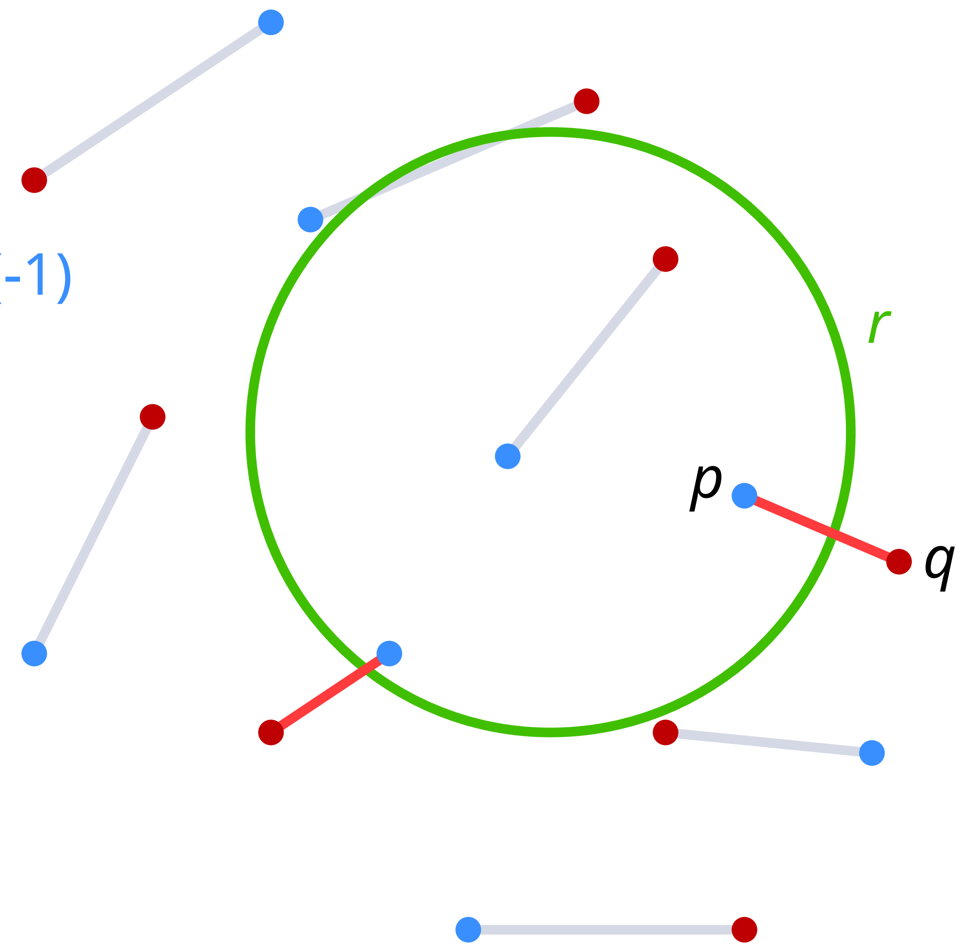

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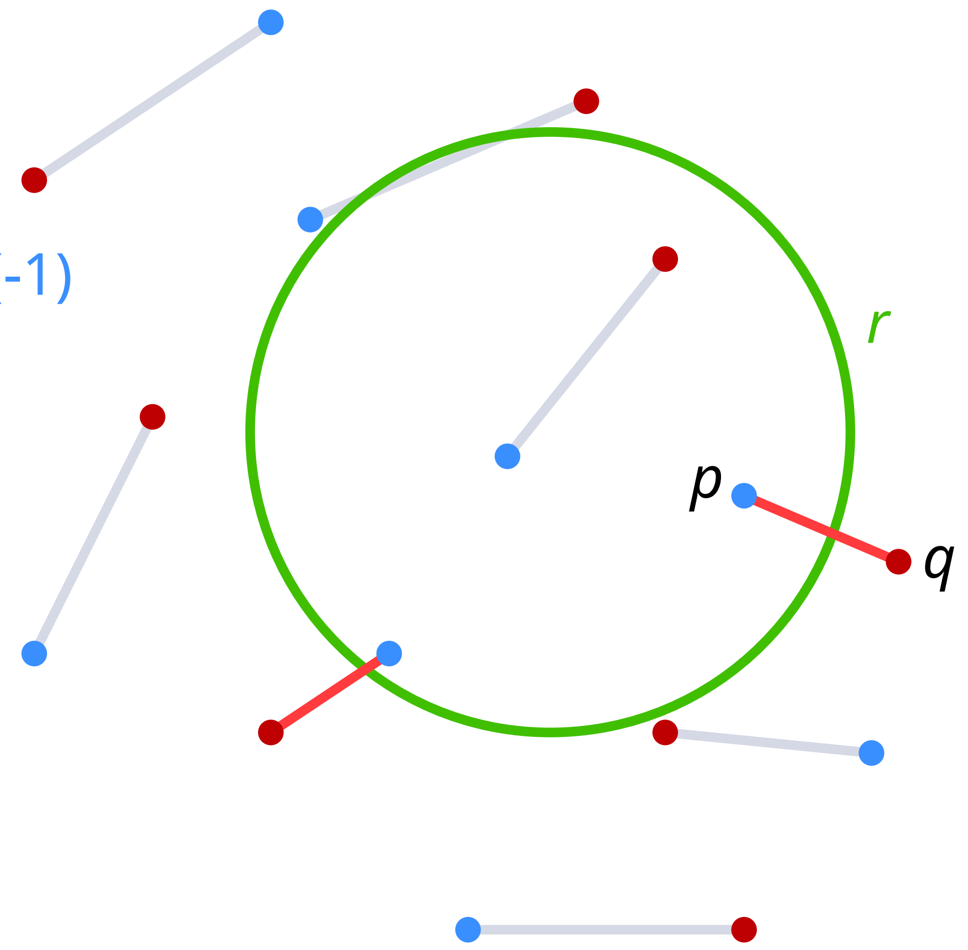
$P[|\chi(r)| > \Delta_r] < \frac{1}{2m}$



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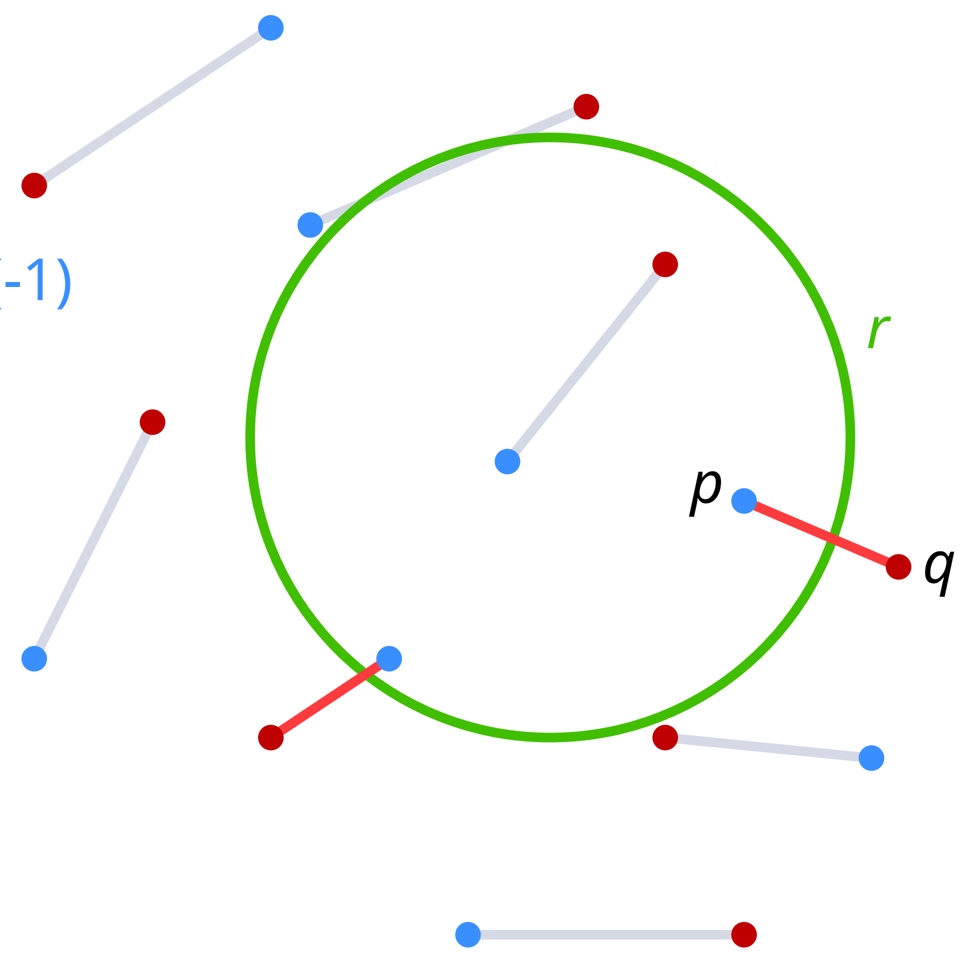

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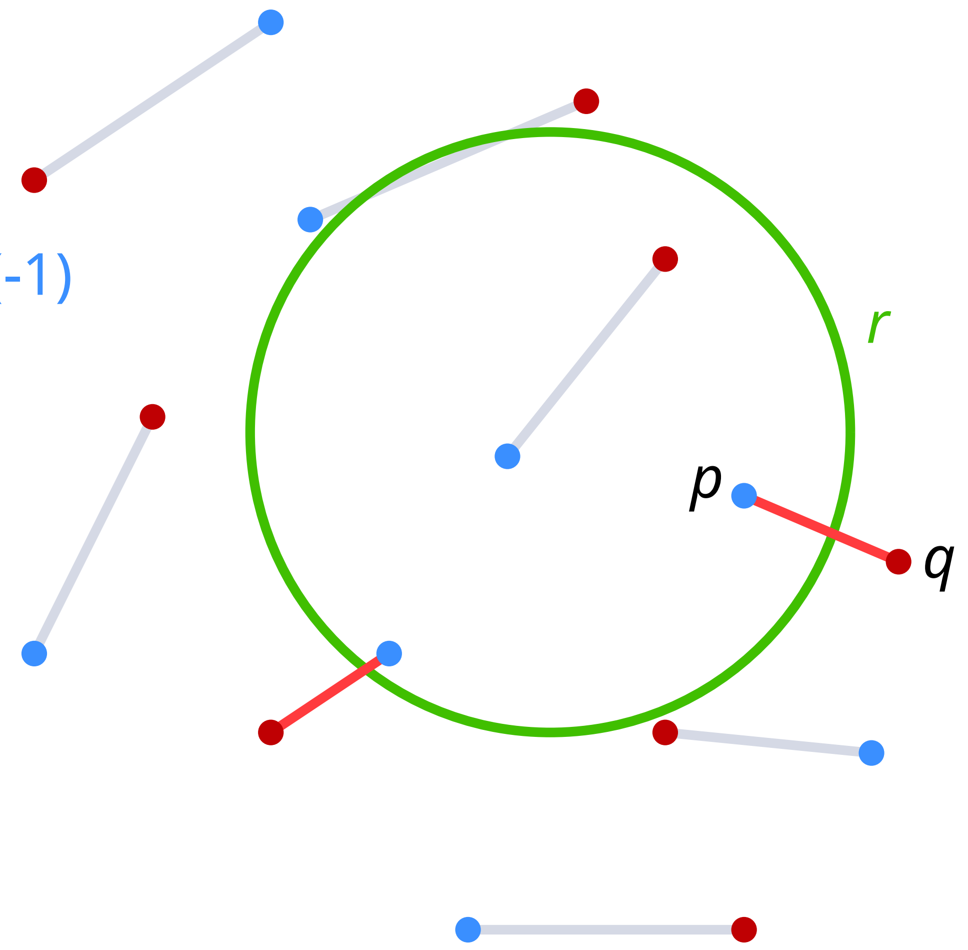
$\Delta_r = O(\sqrt{\delta n \log n})$ for shattering dim. δ since $\#_r \leq n/2$



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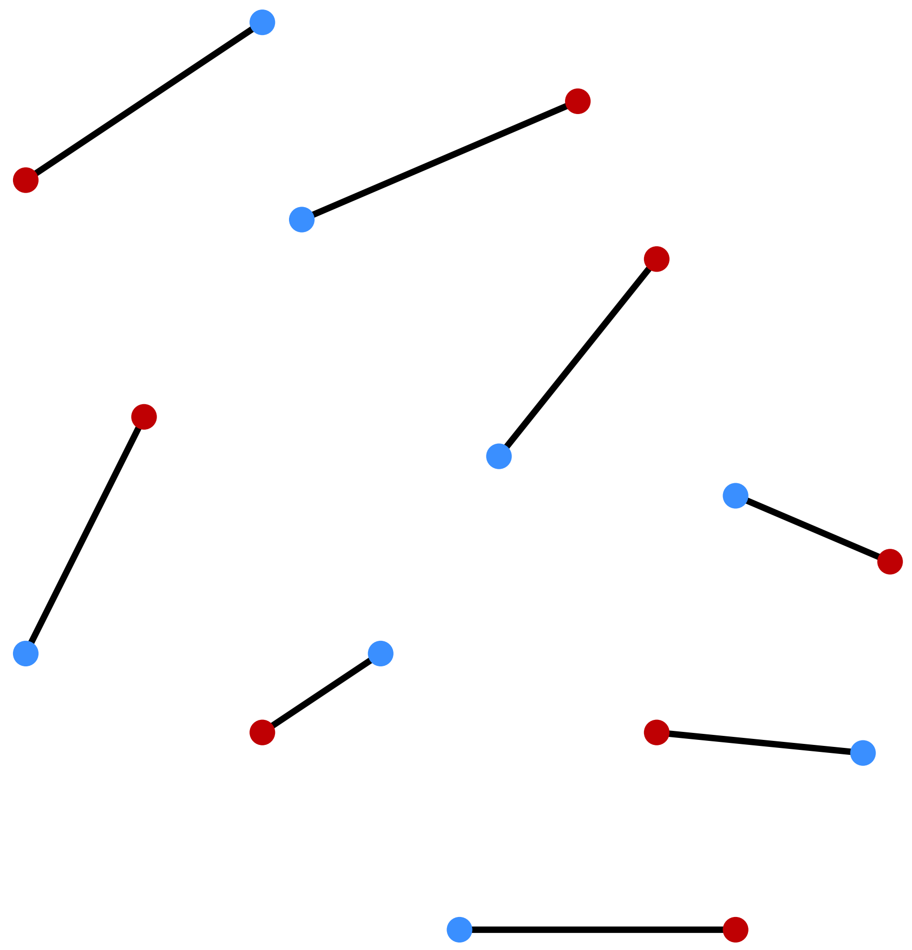
We can compute χ with $|\chi(r)| = O(\sqrt{\delta \#_r \log n})$



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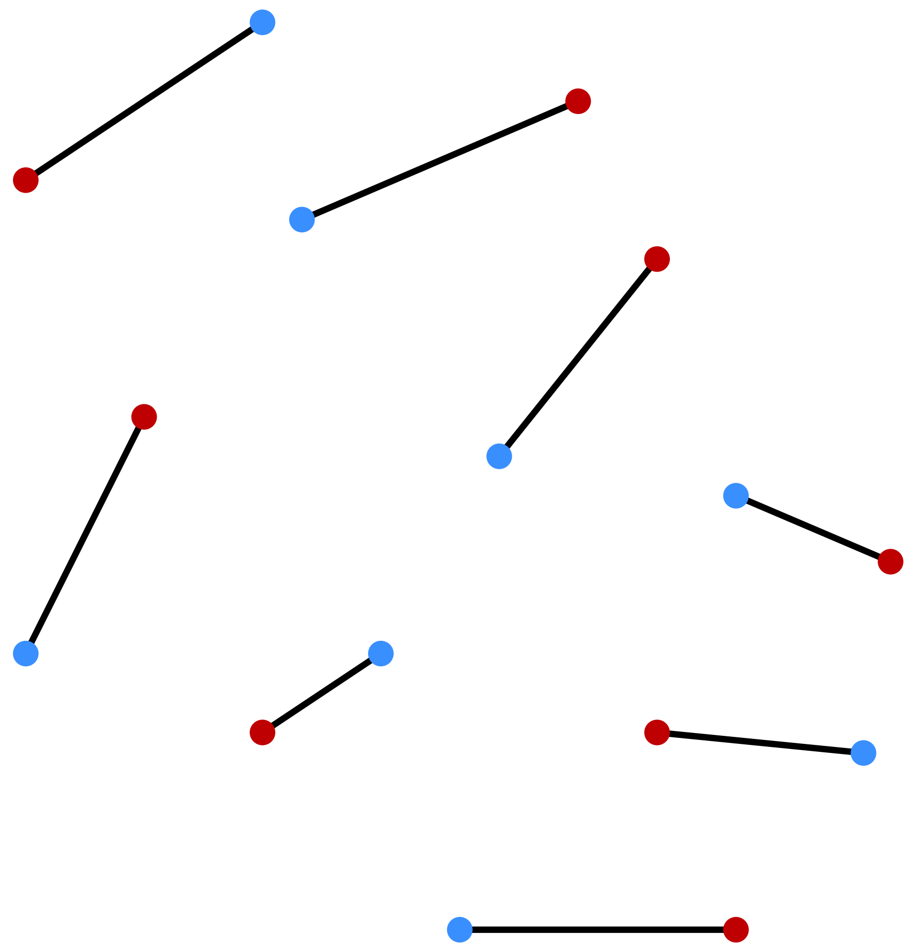

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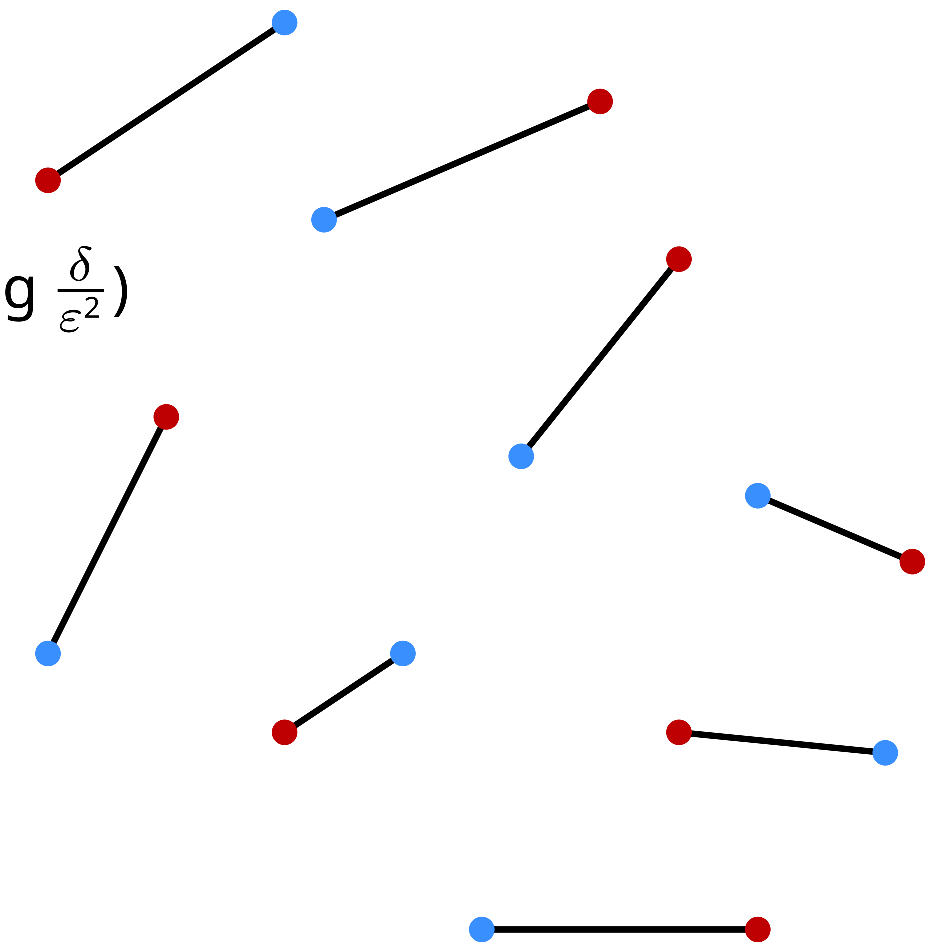

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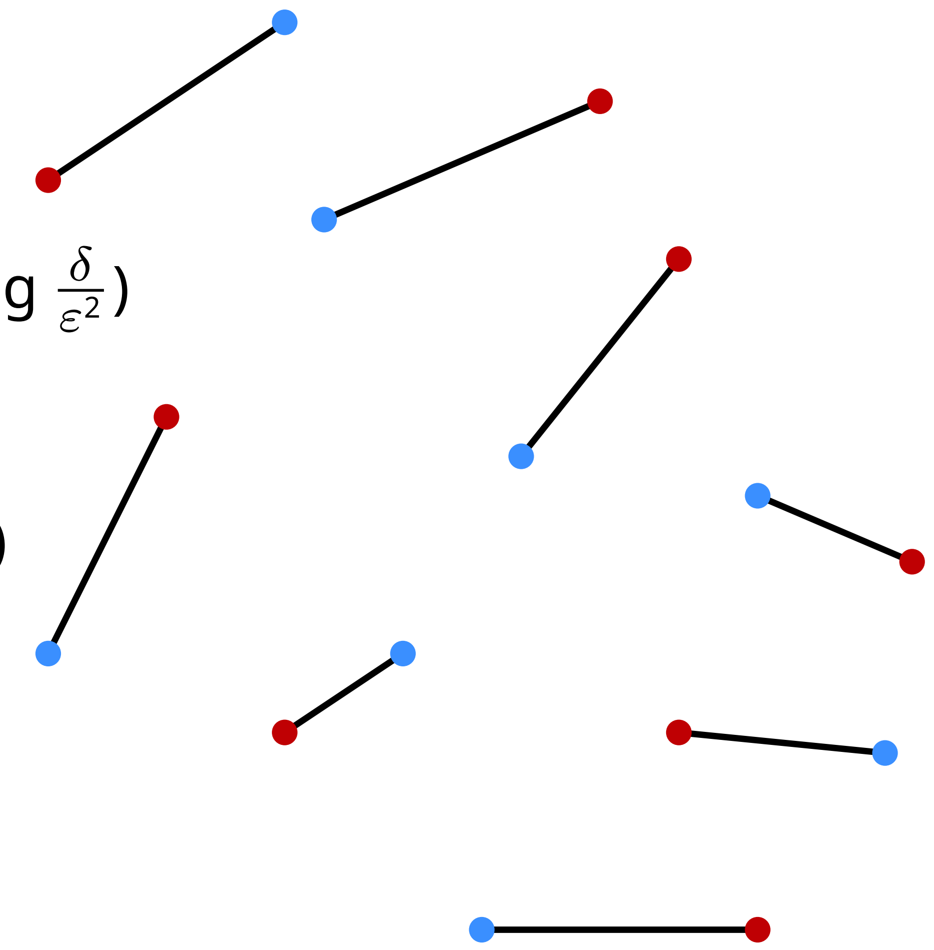
How to improve:

Construct perfect matching, such that $\#_r = O(n^{1-\lambda})$ for some suitable $\lambda > 0$



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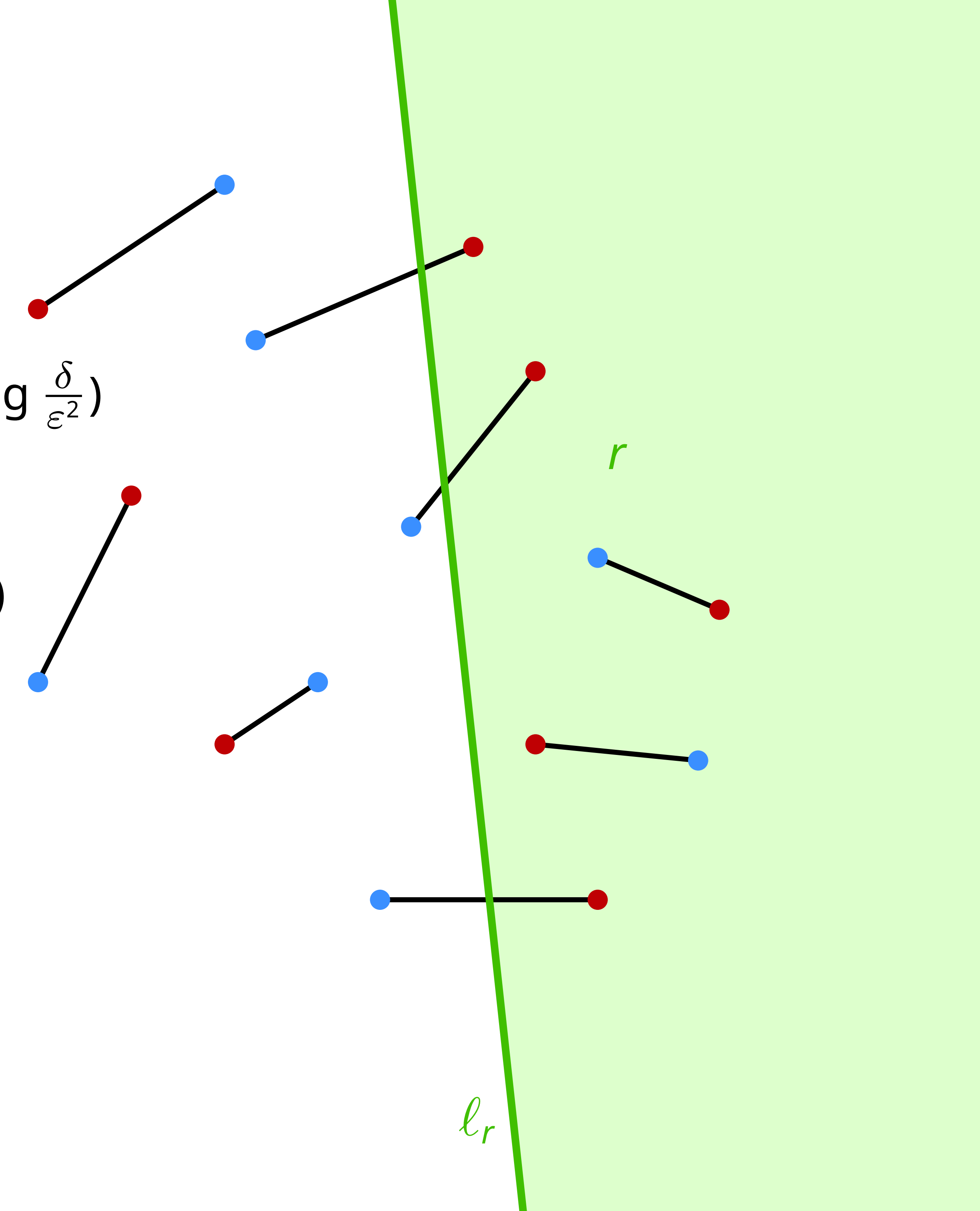
As example, we will consider $\mathcal{R}$ = set of halfspaces



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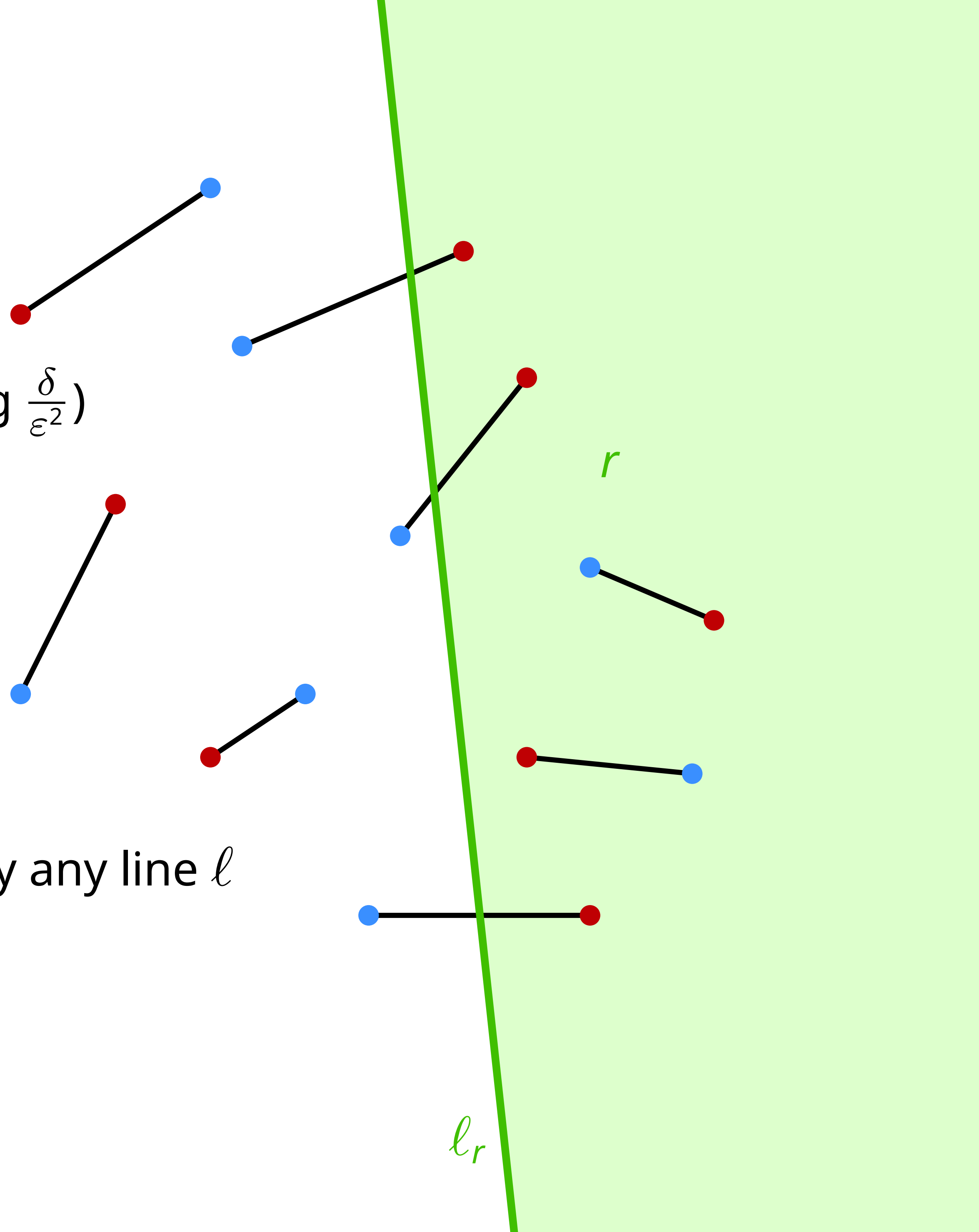
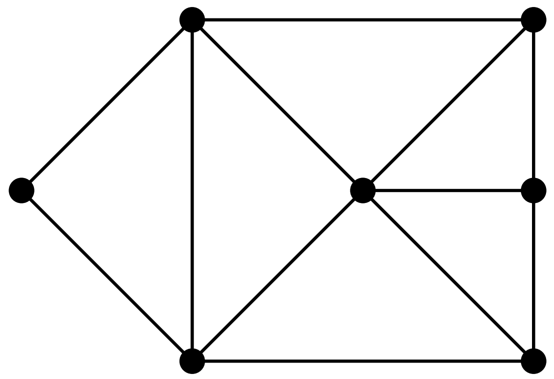
$\max_{r \in \mathcal{R}} \#_r$ = maximum number of edges crossed by any line ℓ



Computing spanning trees with low crossing number

Spanning Tree in General

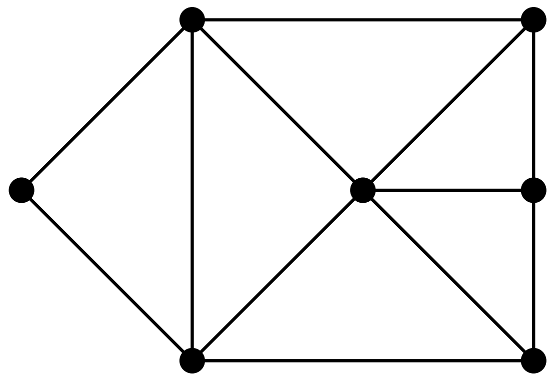
Connected graph $G = (V, E)$



Spanning Tree in General

Connected graph $G = (V, E)$

Spanning tree $T = (V, F)$ of G is a tree with $F \subseteq E$



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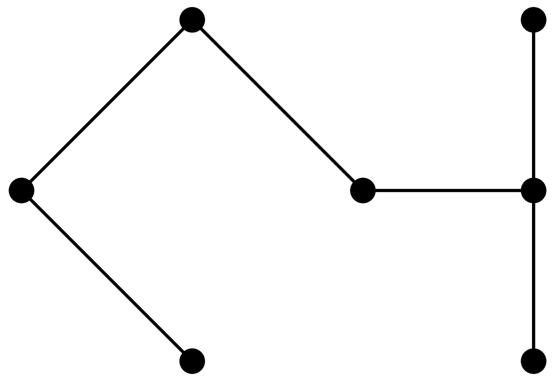

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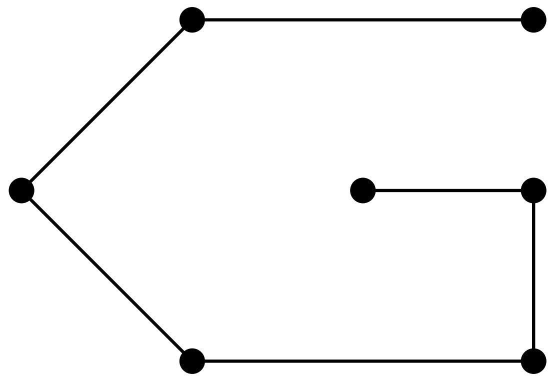

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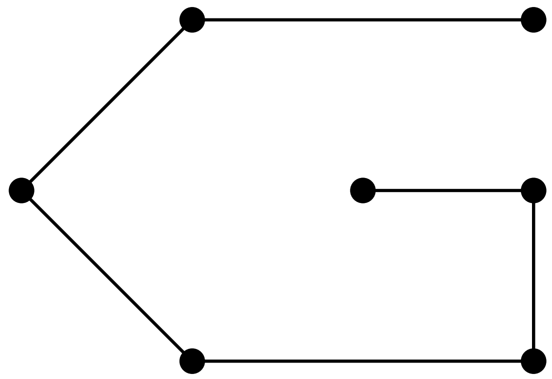

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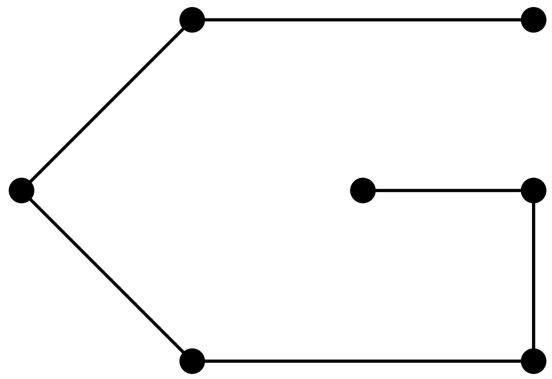
Given edge weights $c : E \to \mathbb{R}_{\geq 0}$

What is the minimum weight spanning tree?



Spanning Tree in the Plane

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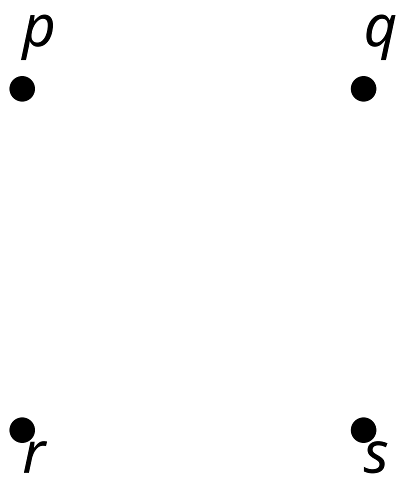

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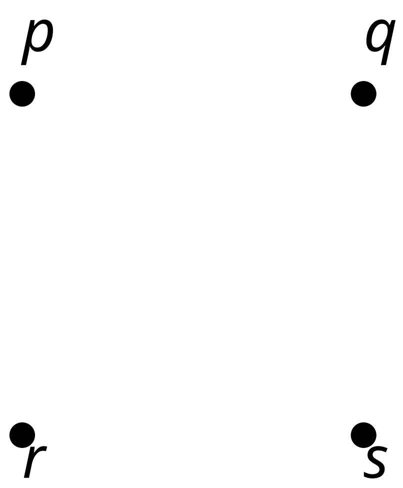
Spanning tree $\mathcal{T}$ are $n - 1$ line segments that span P



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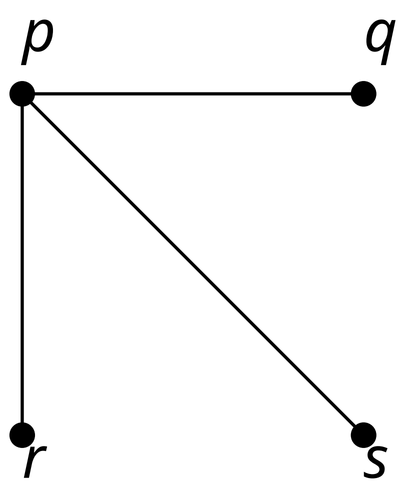

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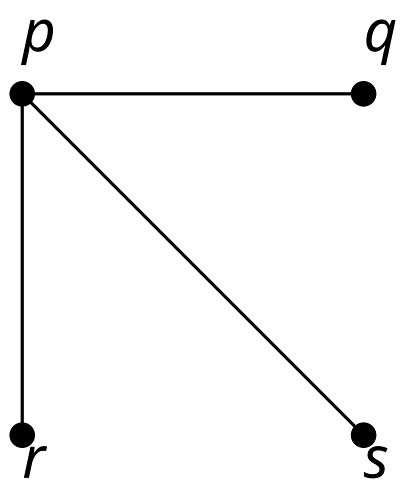

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Quiz Number of distinct spanning trees?

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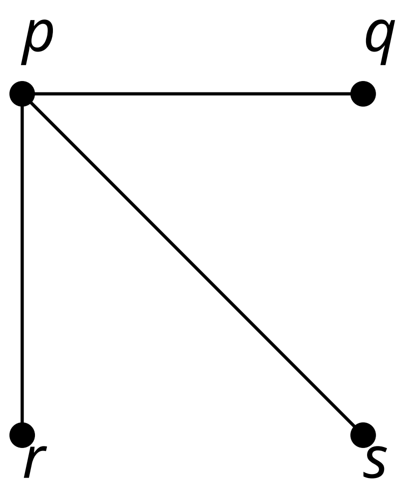
B 12

C 16



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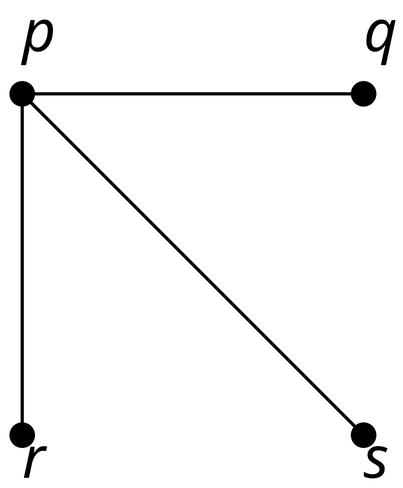
16: from 5 rotatable variations



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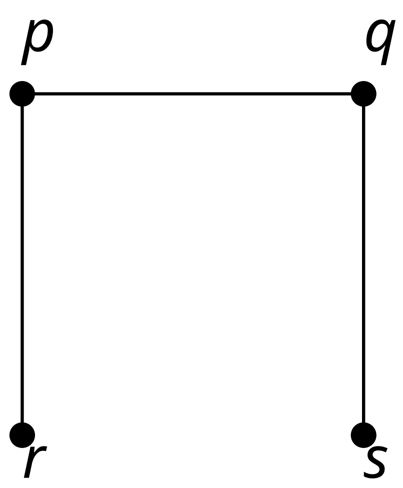

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Set $P \subseteq \mathbb{R}^2$ of n points

Spanning tree $\mathcal{T}$ are $n - 1$ line segments that span P

$\mathcal{T} = \{pq, pr, ps\}$
Note that $pq = qp$

Quiz Number of distinct spanning trees?

A 8

B 12

C 16

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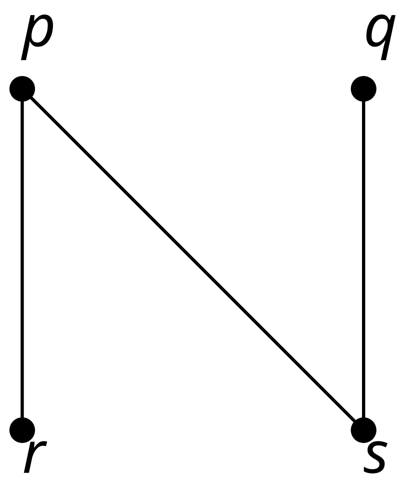

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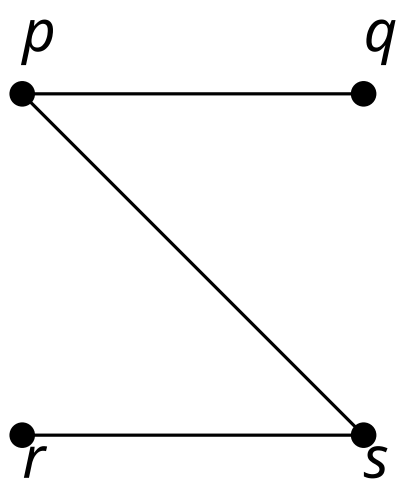

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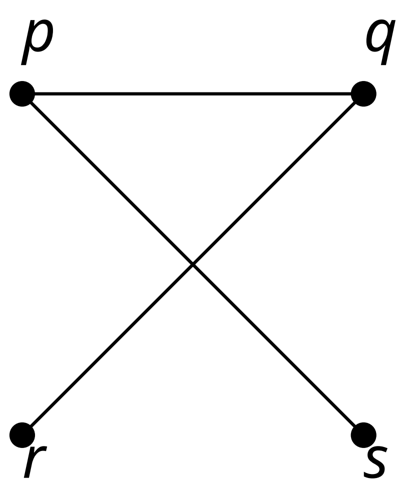

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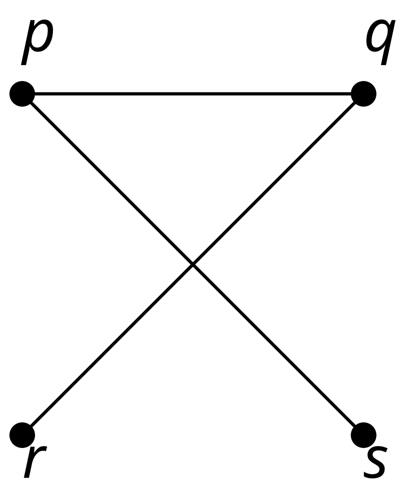

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Cayley's Formula: n^{n-2} trees

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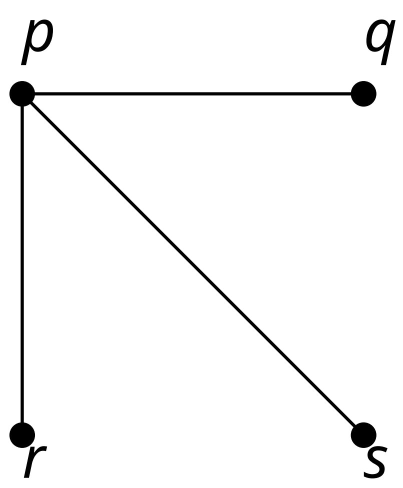

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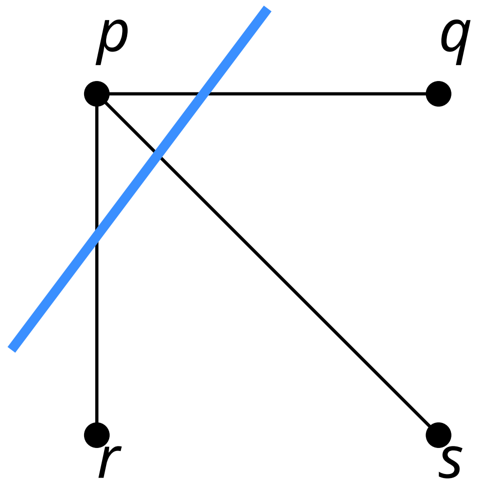

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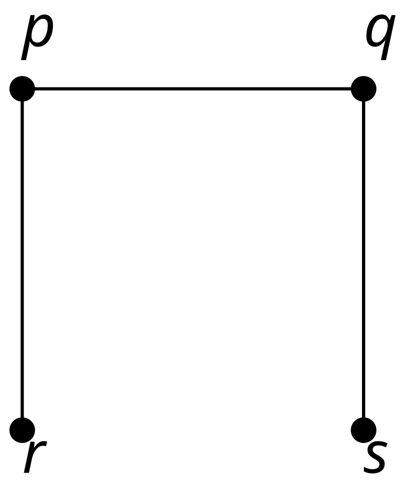

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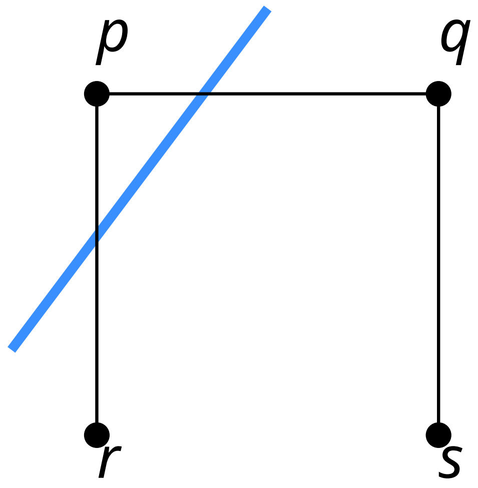

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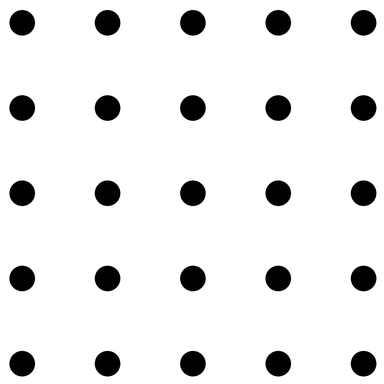
Given n points on $\sqrt{n} \times \sqrt{n}$ grid



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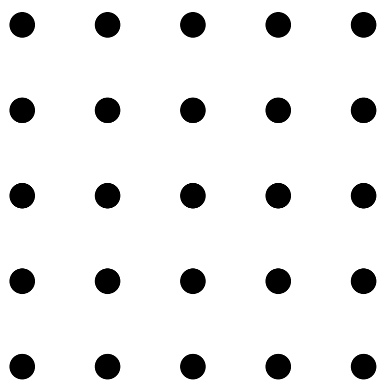

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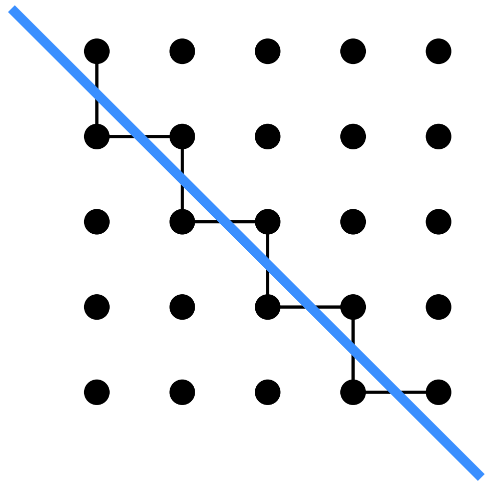

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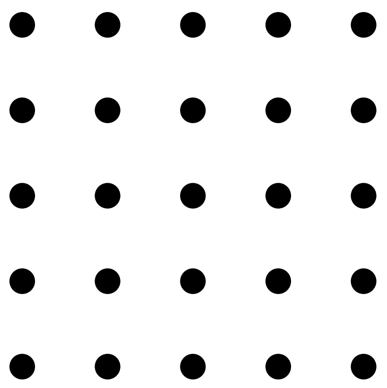

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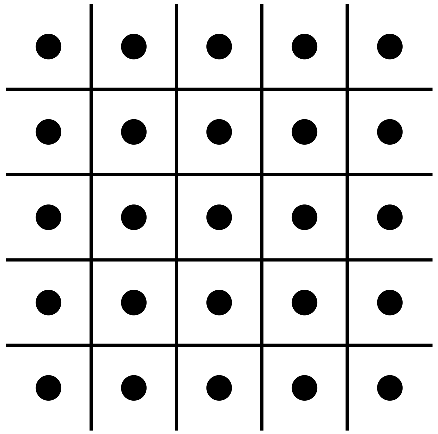

Lower bound (Ω)?

Draw $2 \cdot (\sqrt{n} - 1)$ lines

Each line segment crosses at least one line

For at least one line $\frac{n-1}{2 \cdot (\sqrt{n}-1)} = \Omega(\sqrt{n})$ line segment crossings (pigeonhole principle)

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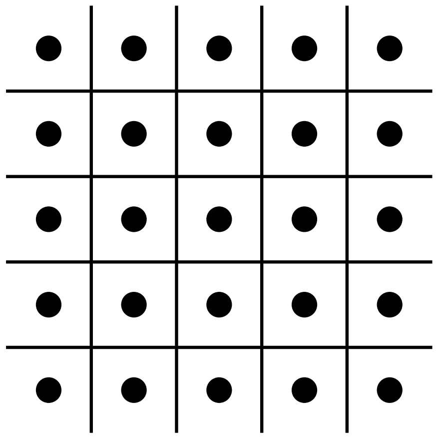

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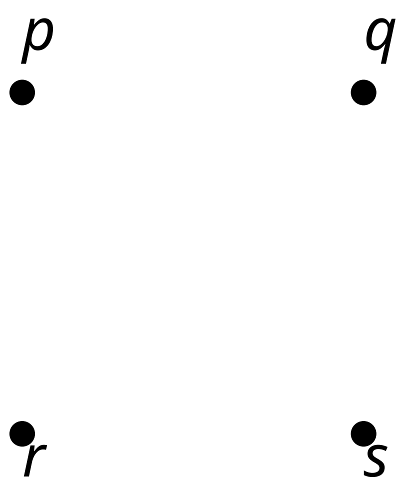
Theorem. We can always find $\mathcal{T}$ with stabbing number $O(\sqrt{n})$ in polynomial time (1992 Welzl)

The Algorithm

Consider all separating lines $\hat{L}$ of P



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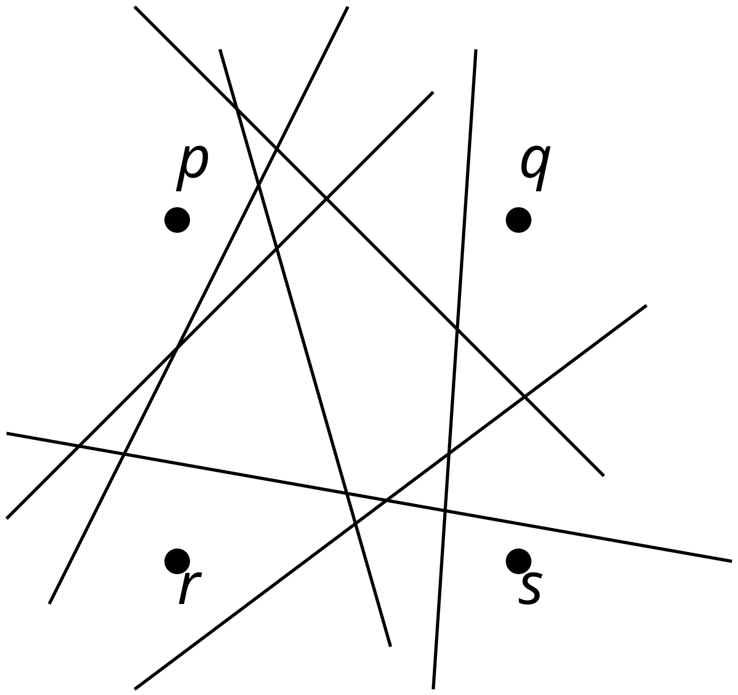

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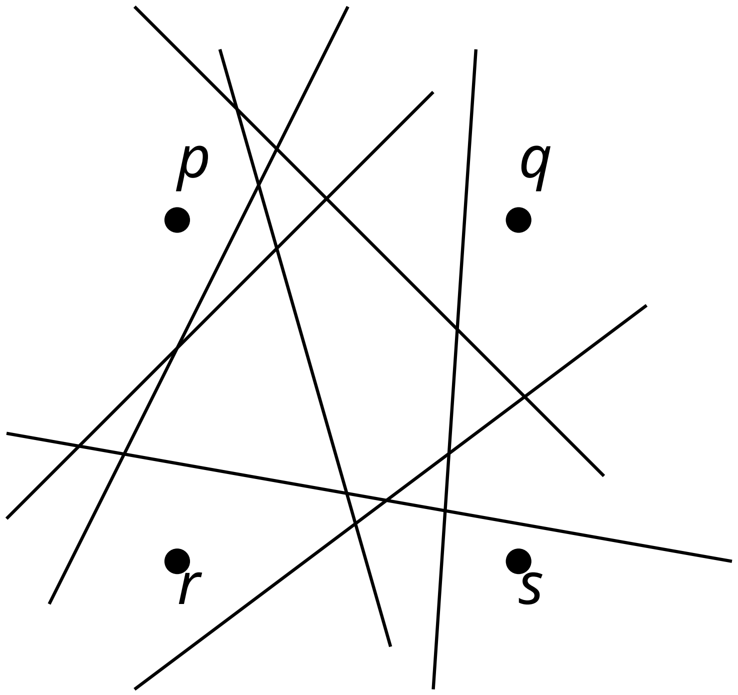

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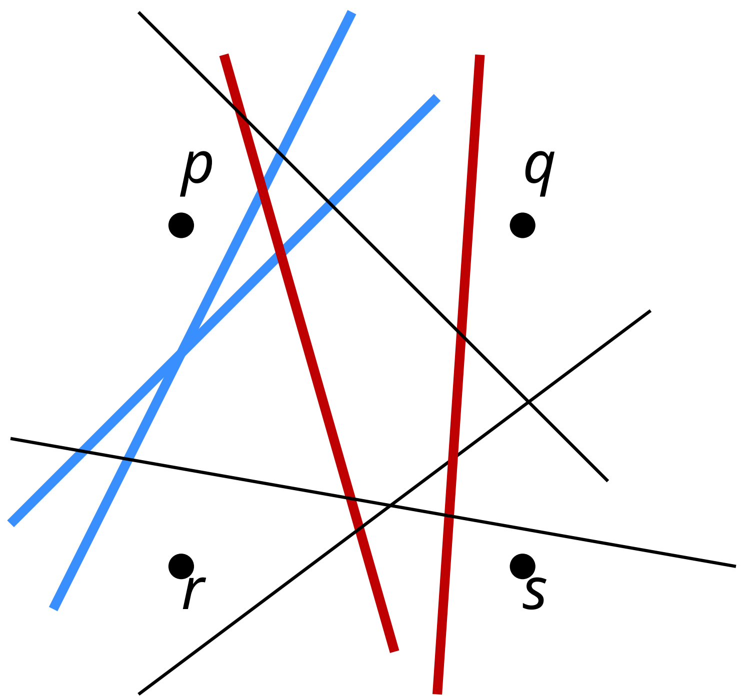

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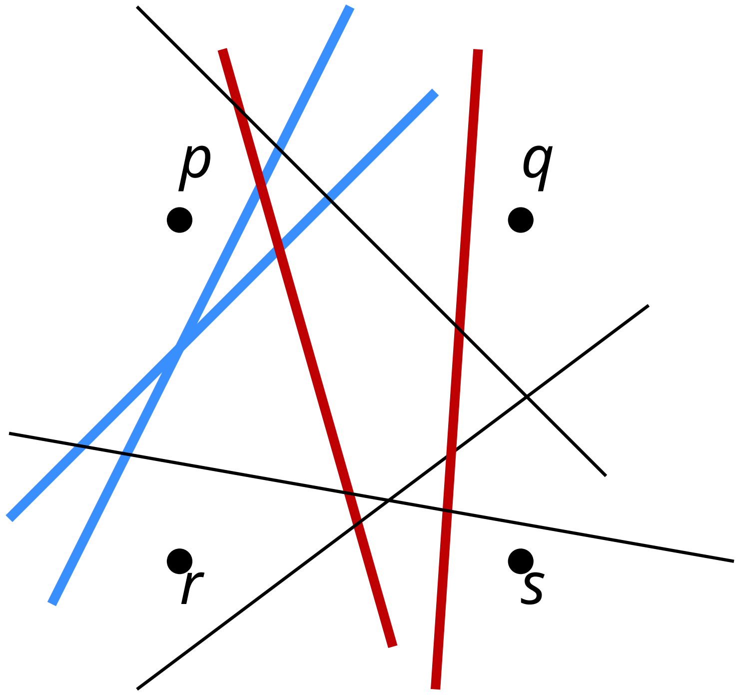
Pick one for each equivalent class of $\hat{L}$
Let this be set L



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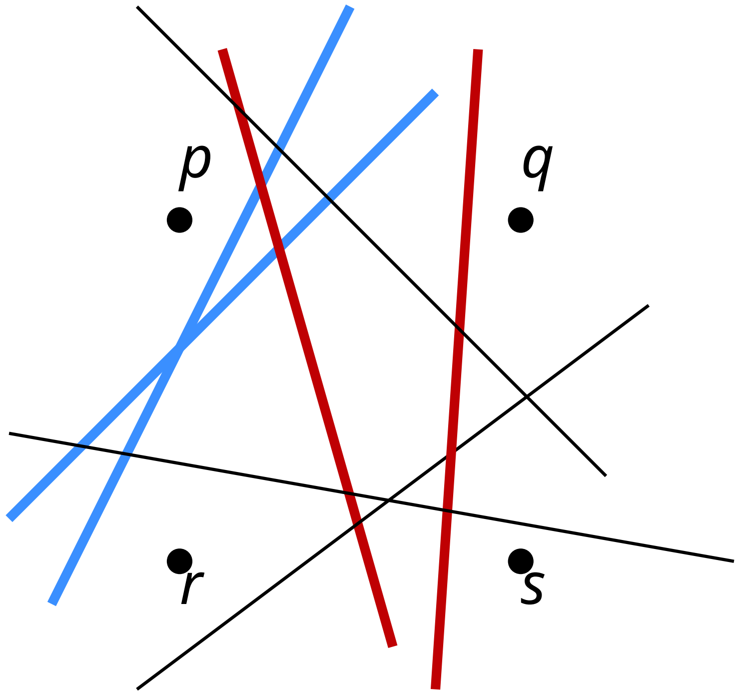

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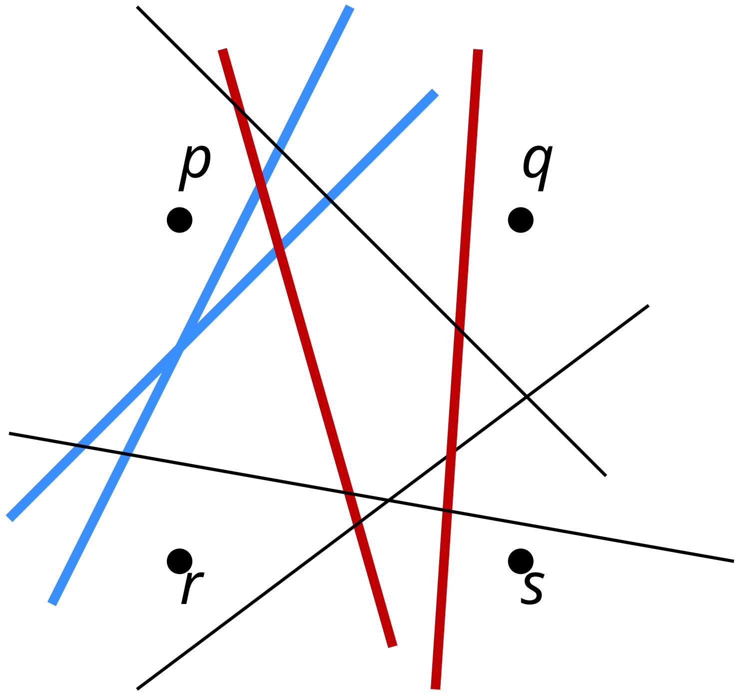

$|L| \leq 4\binom{n}{2}$, rotate every line until it goes through two points

The two points are the same for at most 4 lines ((above, above), (above, below), ...).

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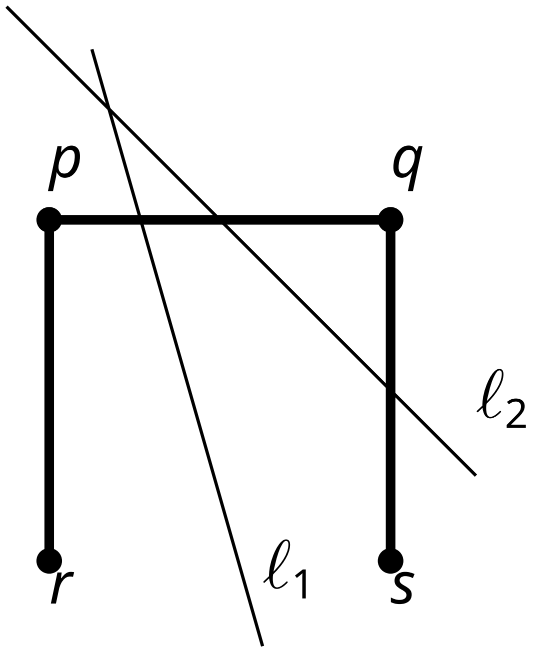

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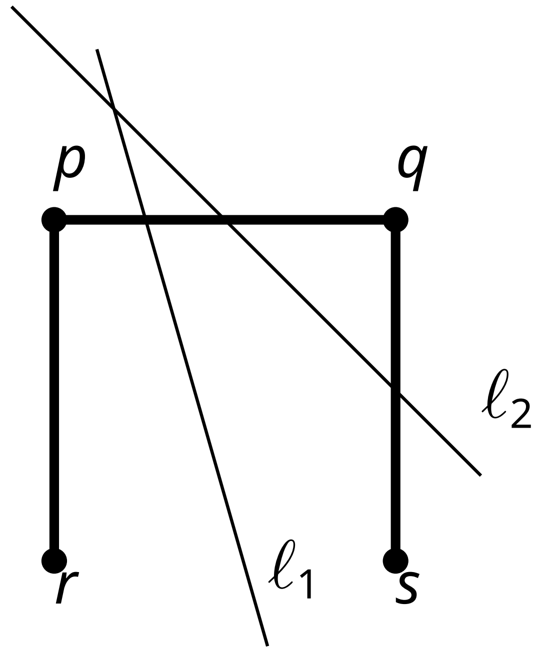

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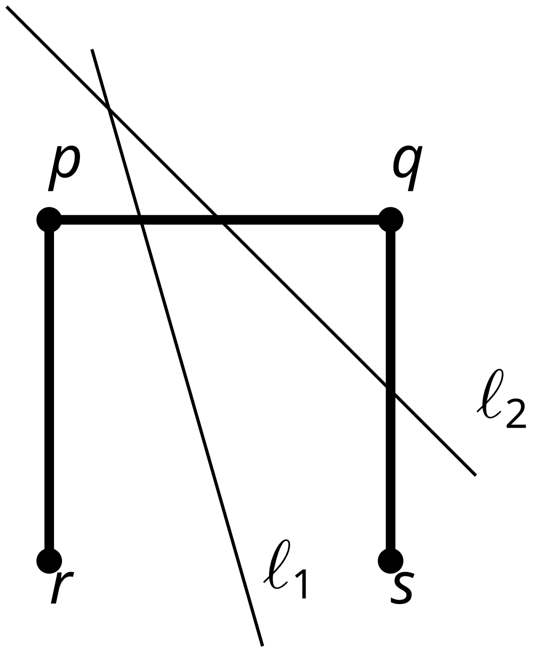

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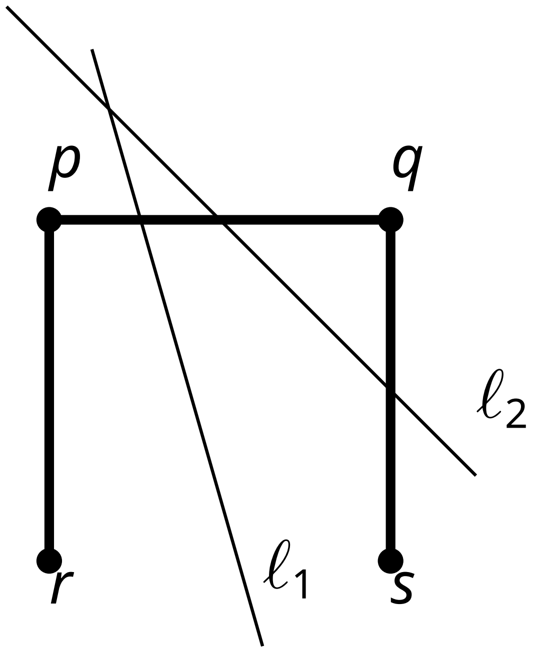

While $|P| > 1$

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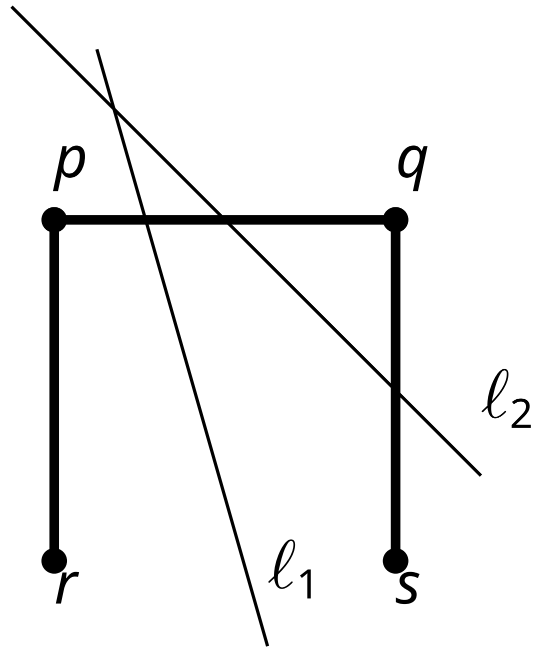

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The running time is polynomial in n

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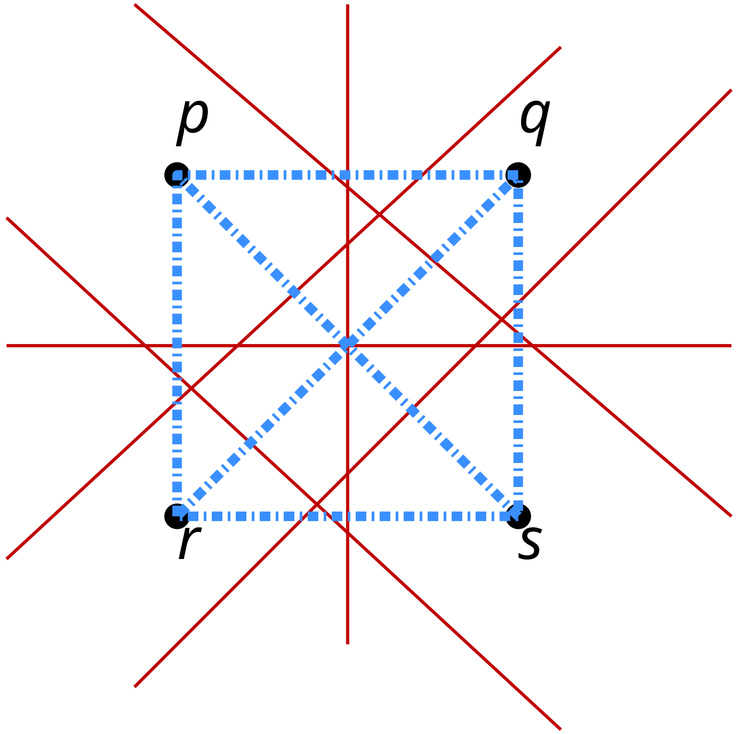

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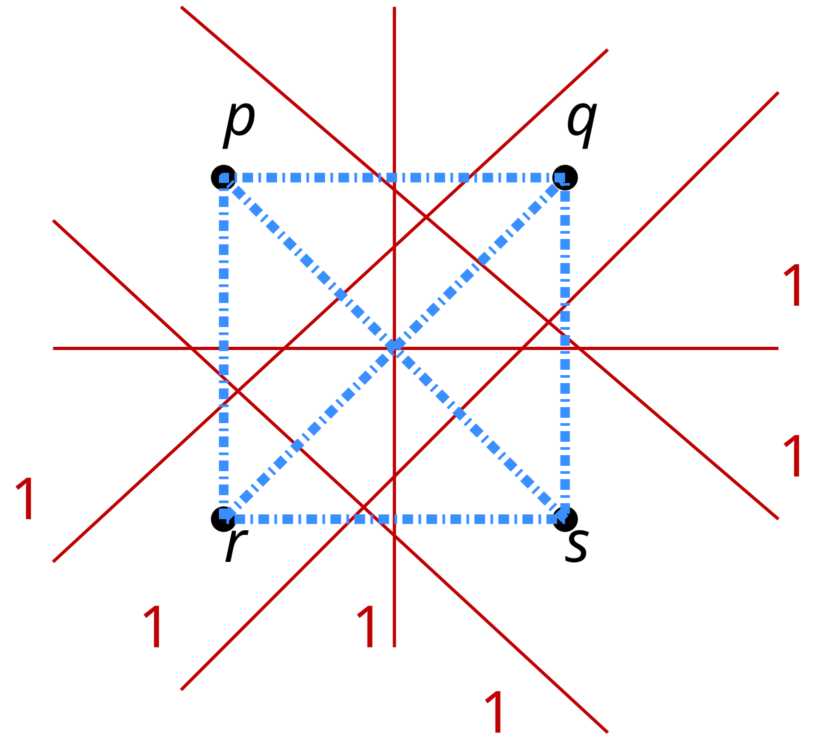

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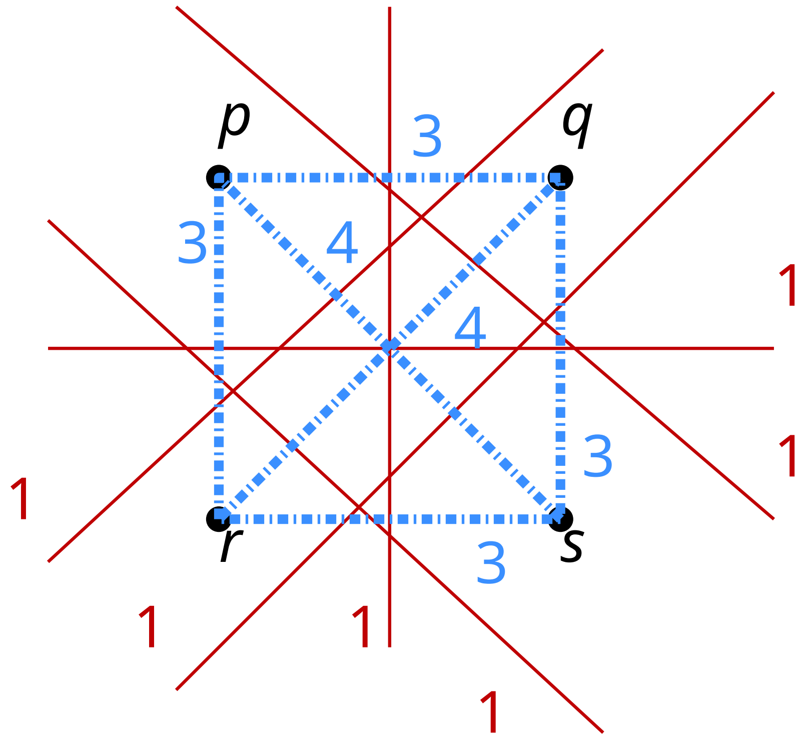

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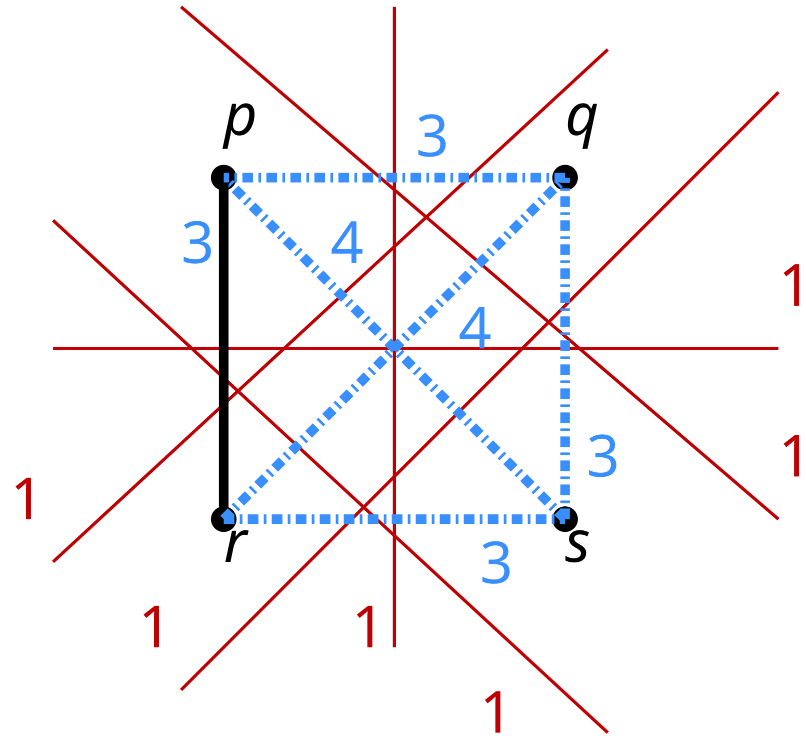

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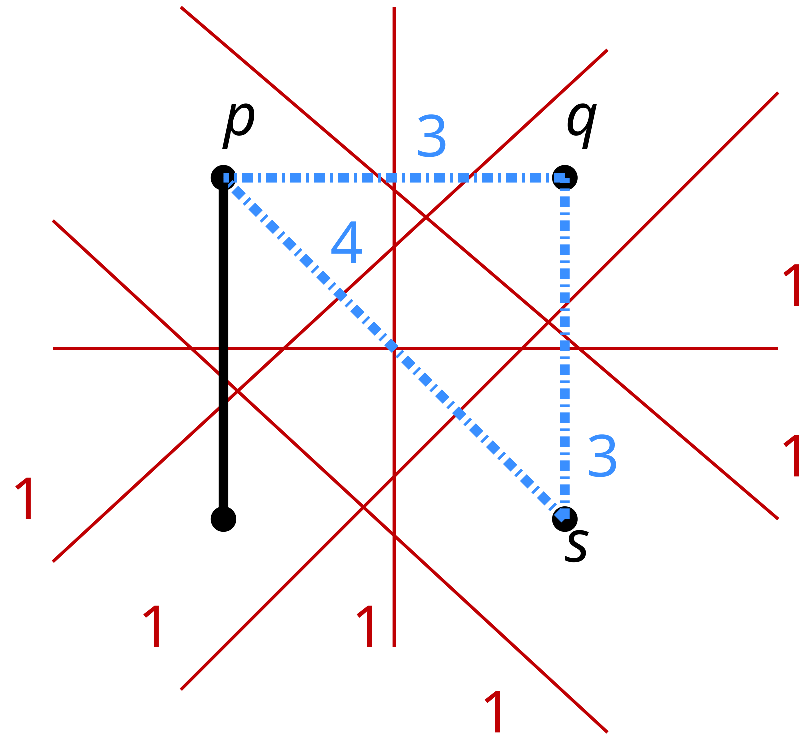

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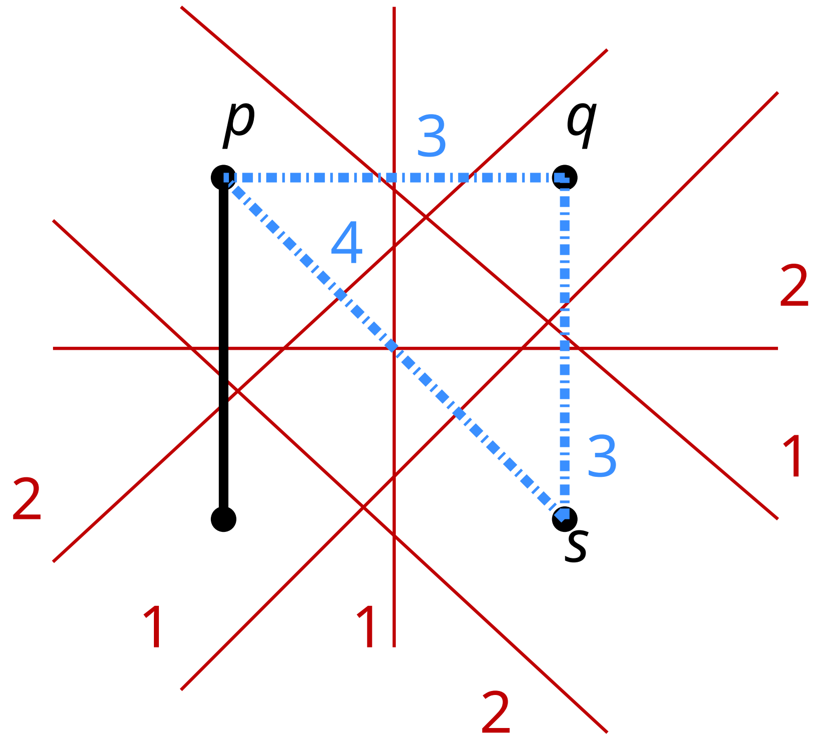

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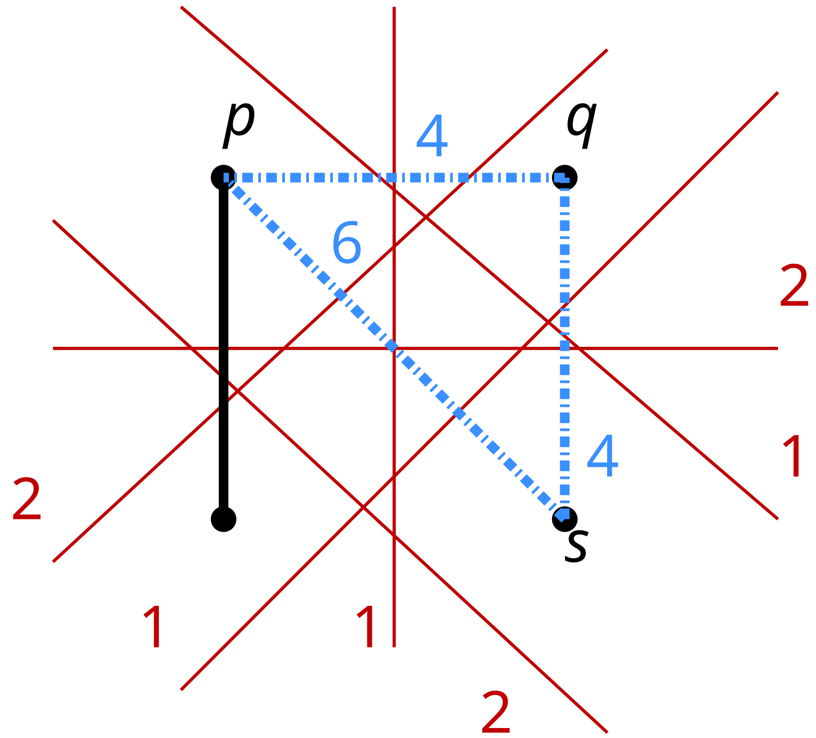

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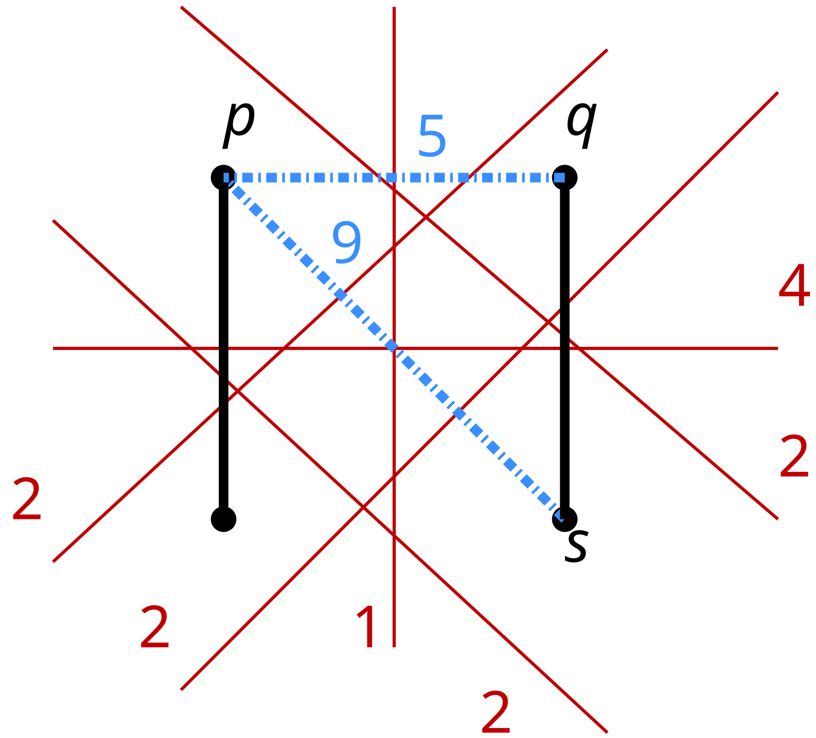

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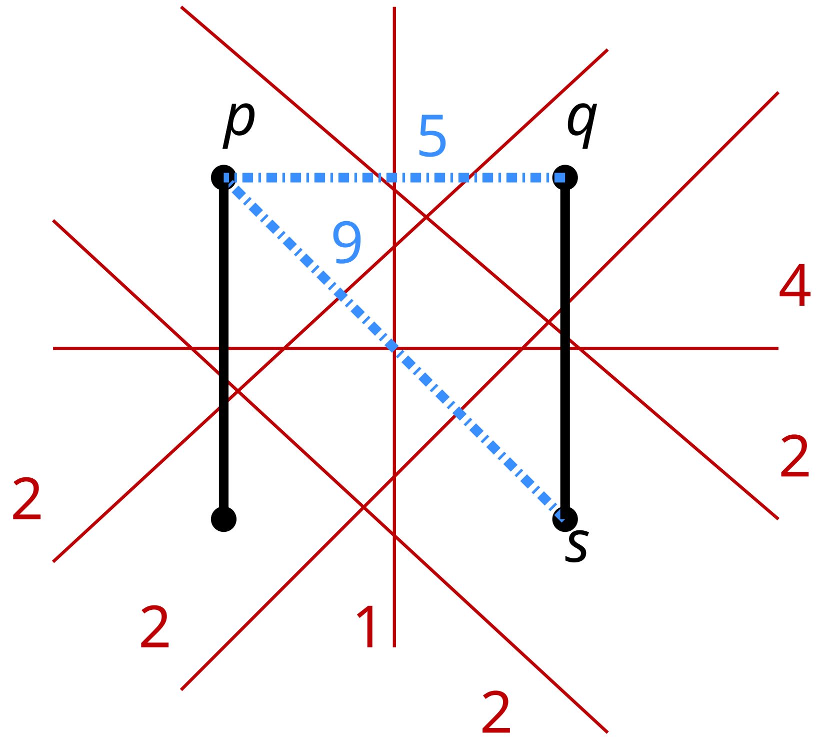
Removing s leads to stabbing number 2

Removing q leads to stabbing number 3

Asymptotically it does not matter



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$d_{✂}(p, q)$ is the crossing distance for $p, q \in P$

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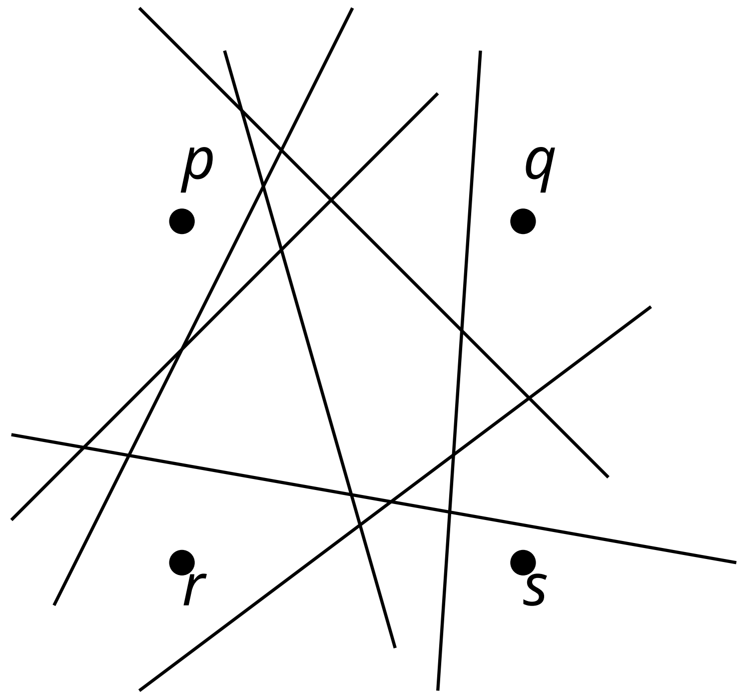

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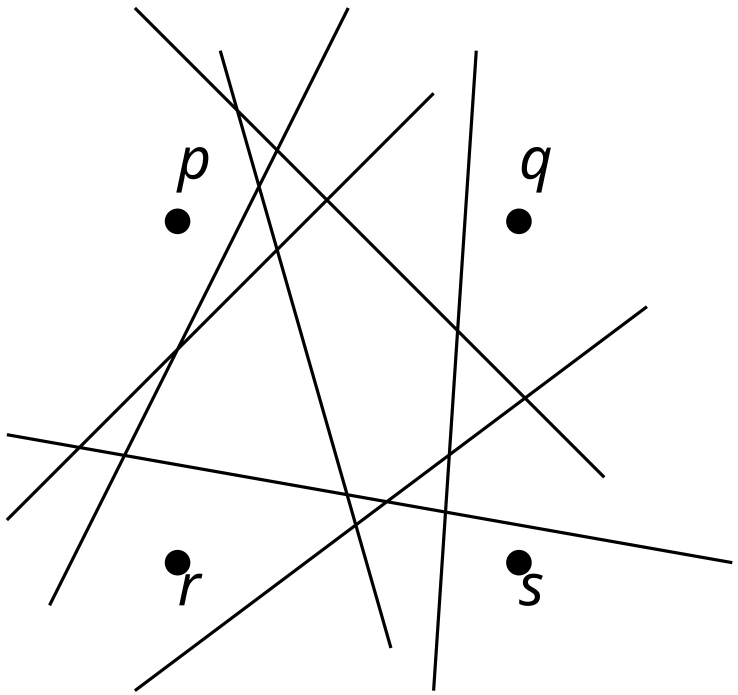

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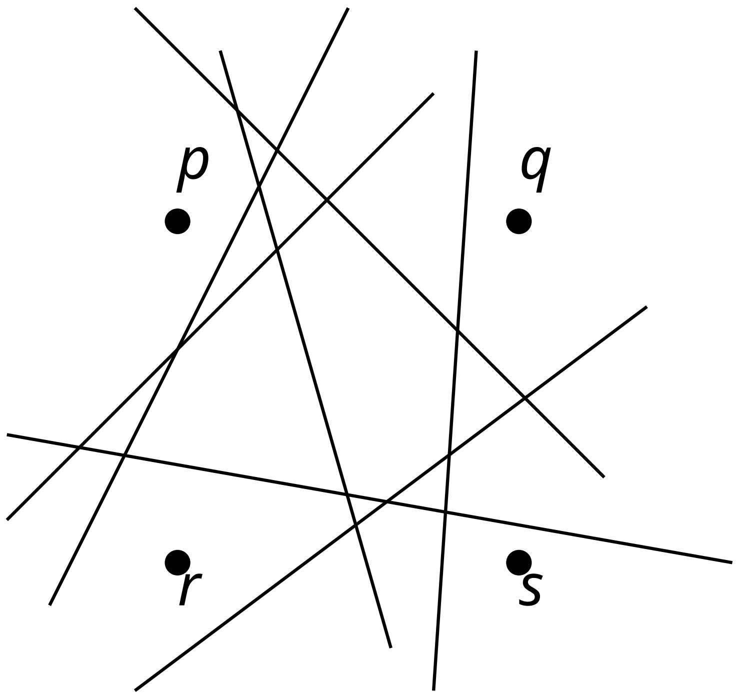

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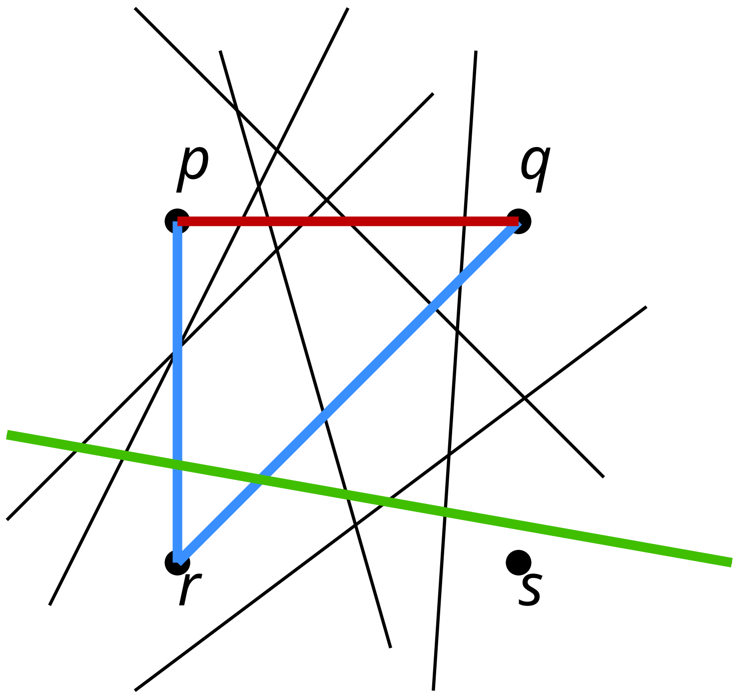

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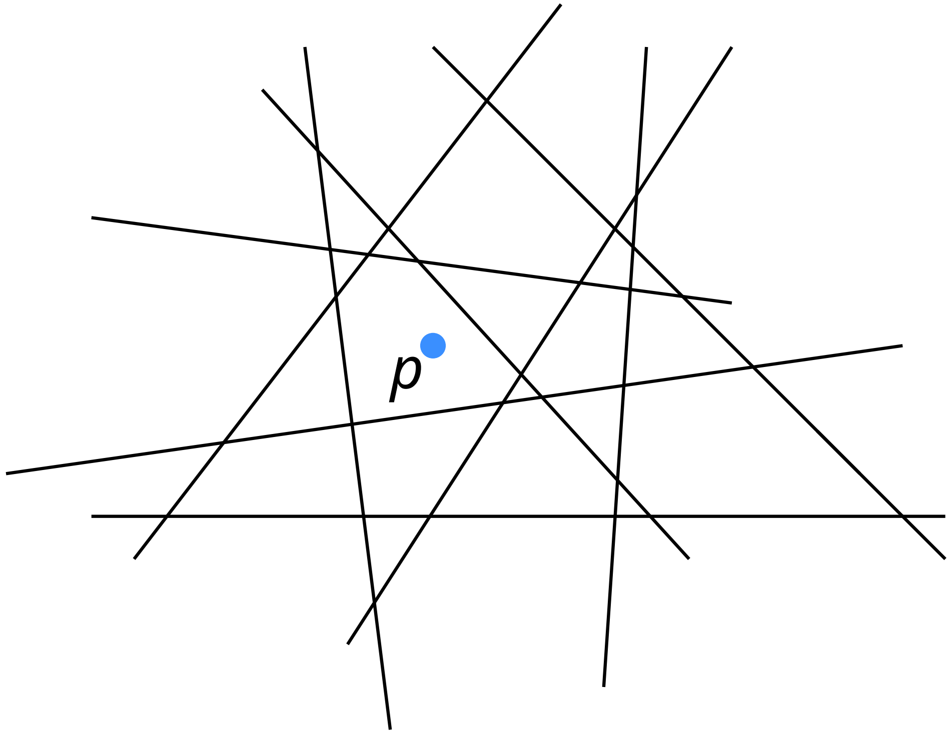

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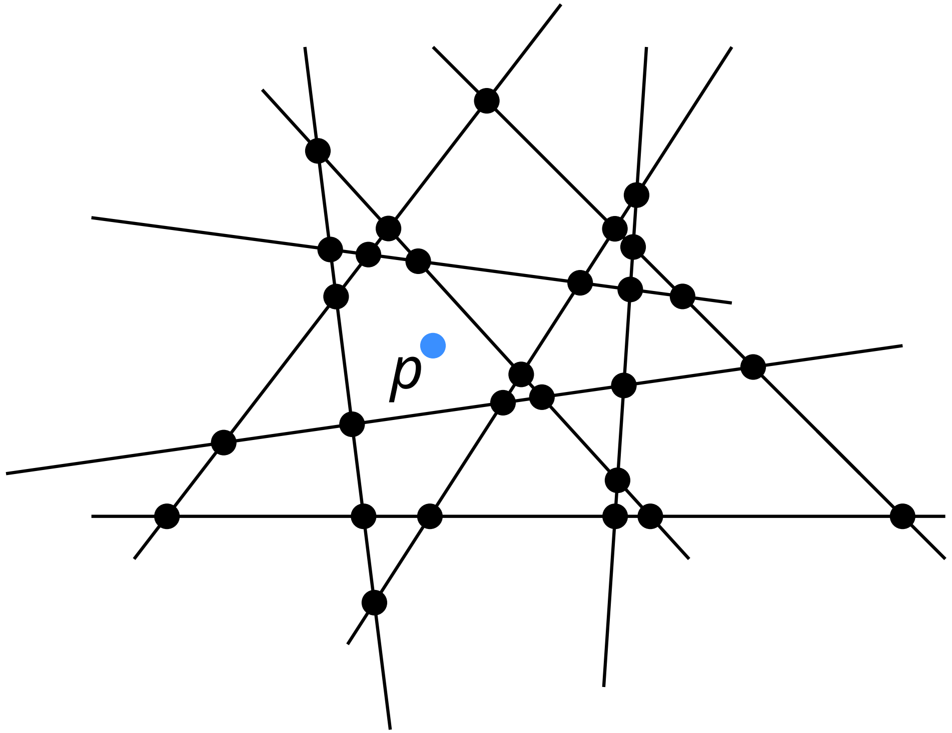
Consider the arrangement $\mathcal{A}(L)$



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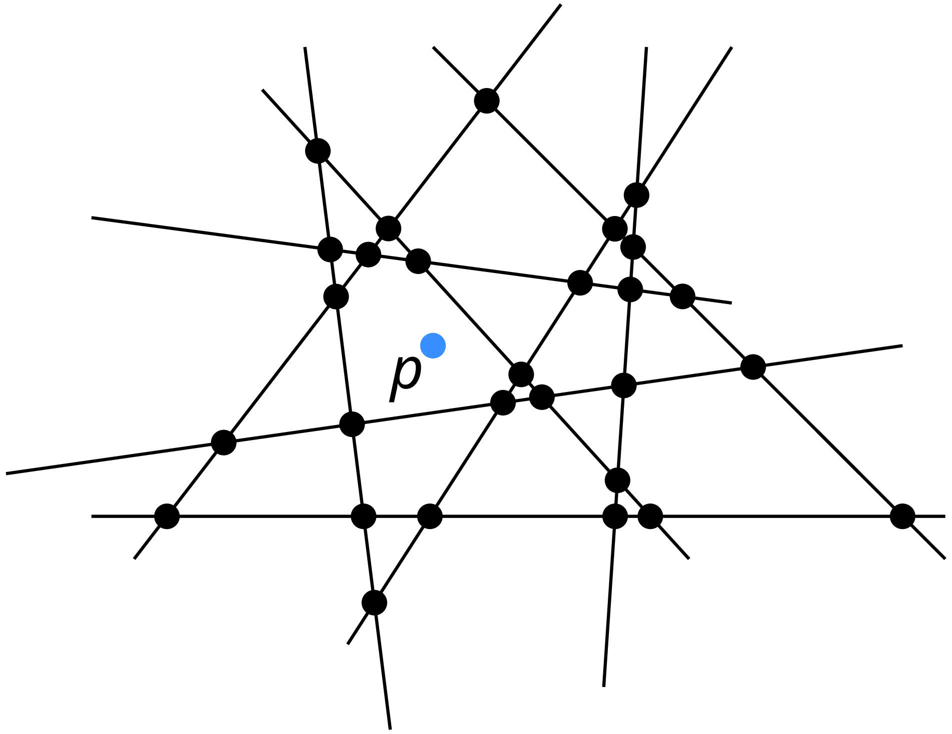
Let $b_{✂}(p, r)$ denote all intersections $q \in \mathcal{A}(L)$ for which $d_{✂}(p, q) \leq r$



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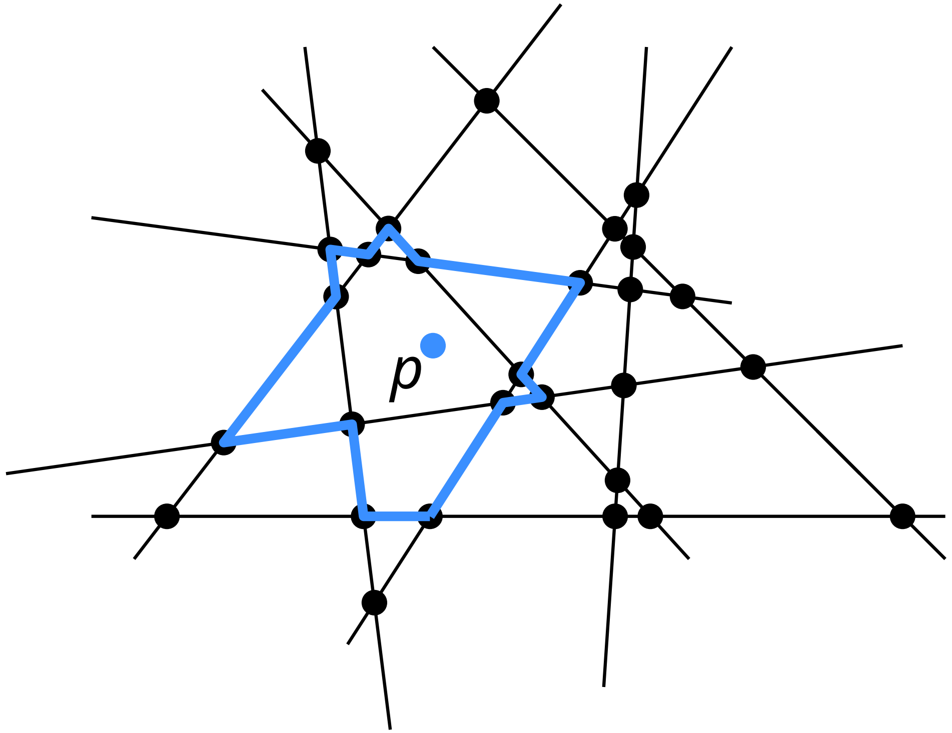

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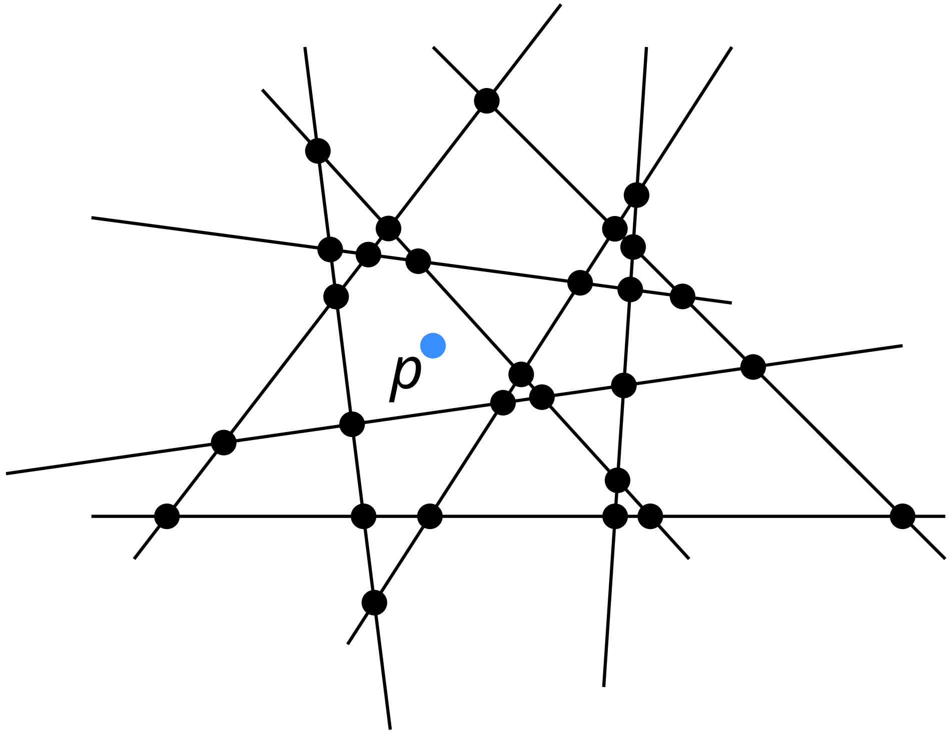

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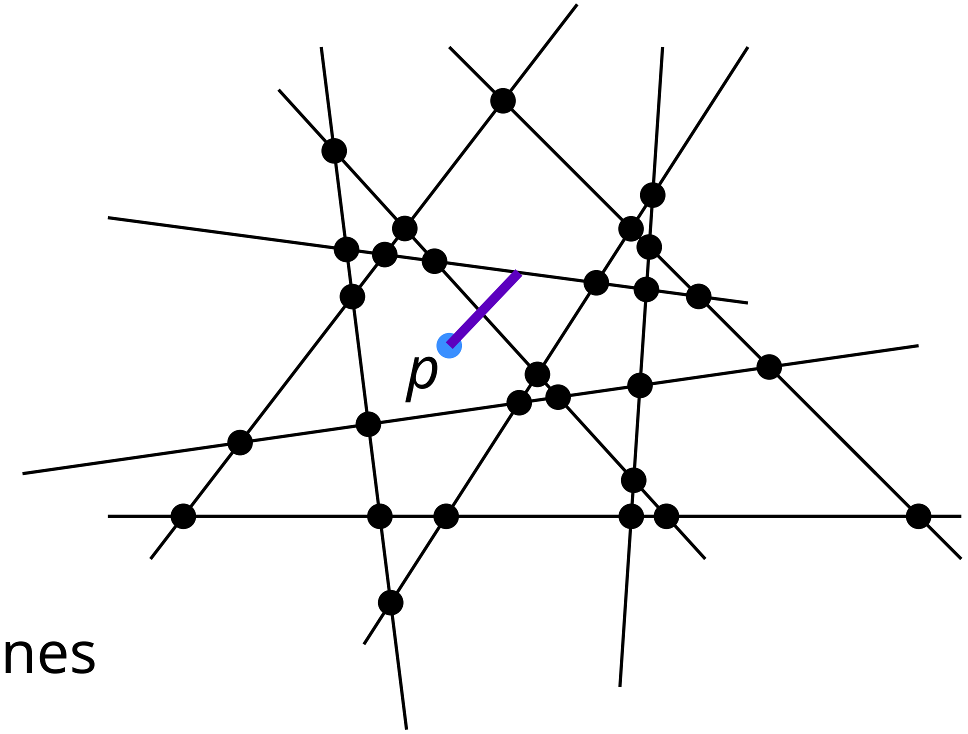

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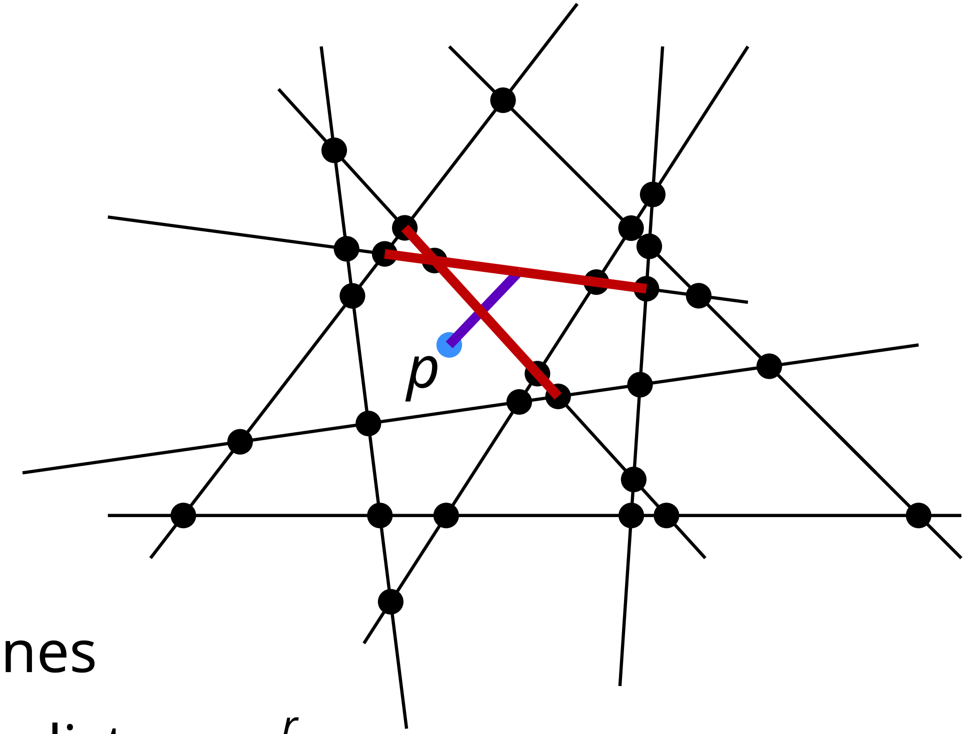

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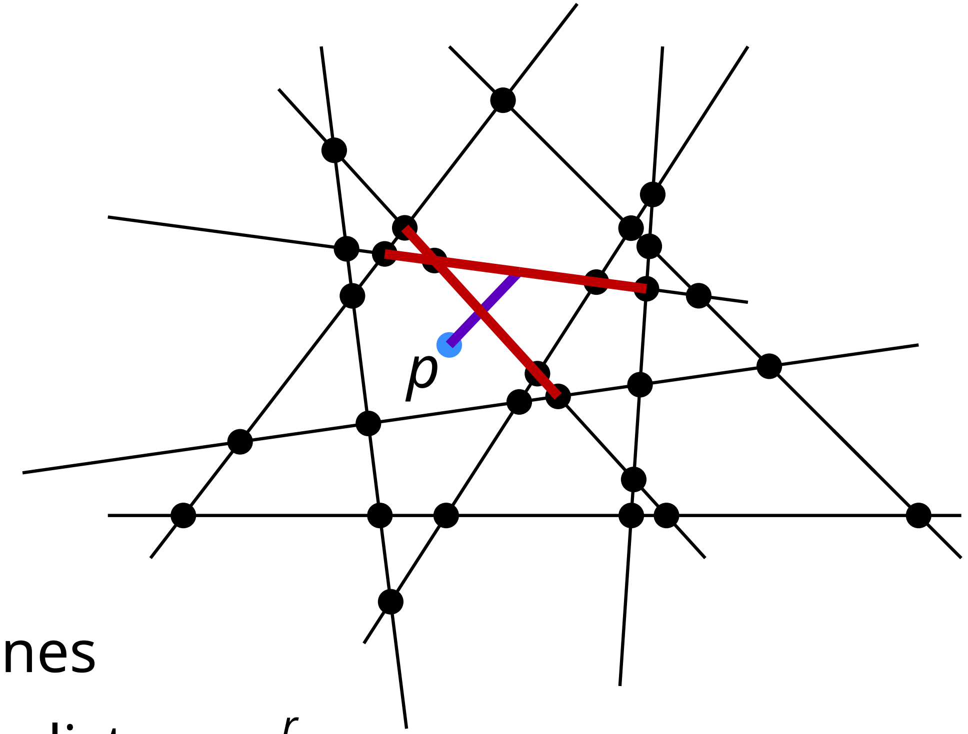

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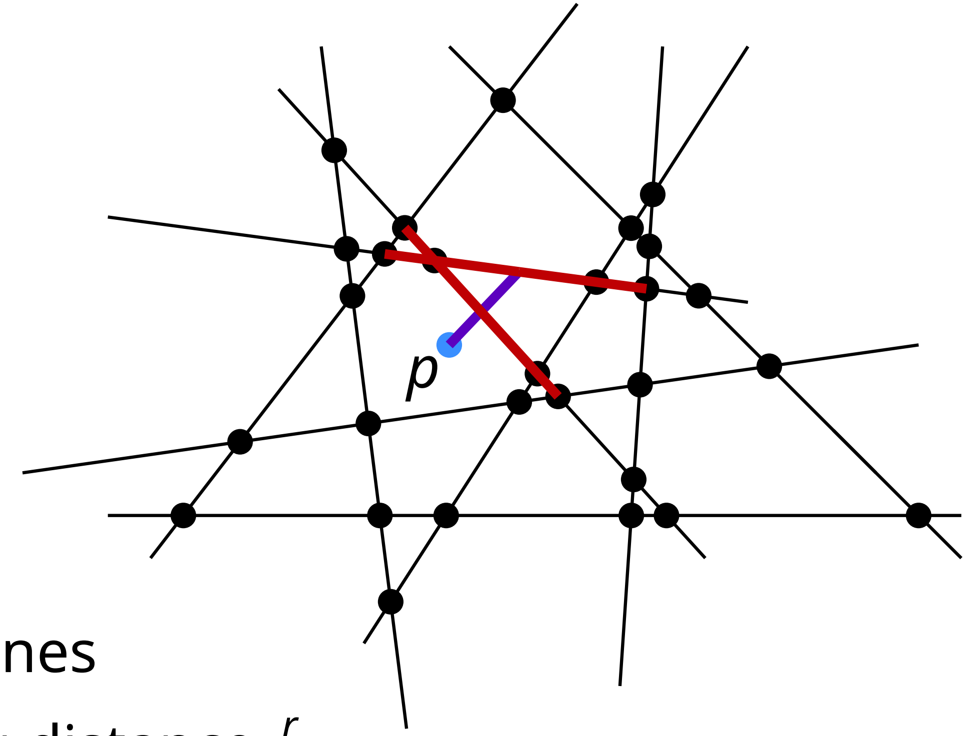

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At least $\frac{r}{2}$ are marked per line and each can be marked at most twice
$\left|b_{✂}(p, r)\right| \geq \frac{r}{2} \cdot \frac{r}{2} \cdot \frac{1}{2} = \frac{r^2}{8}$

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Given $P \subseteq \mathbb{R}^2$ and lines L in the plane with total weight W

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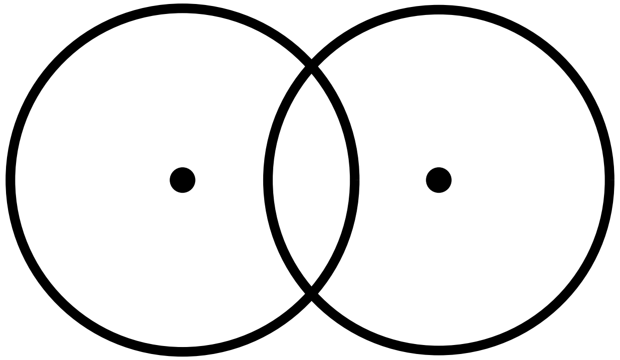

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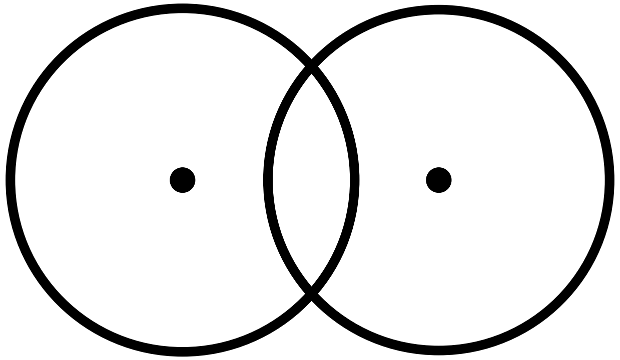

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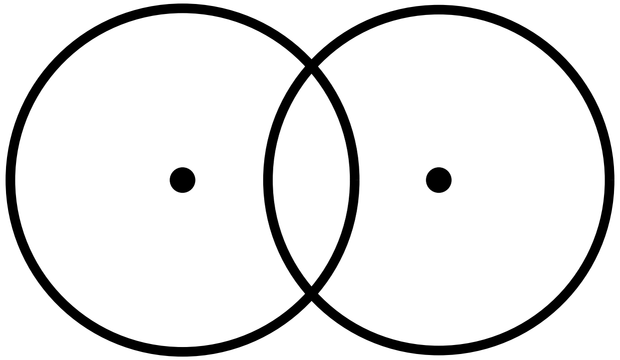

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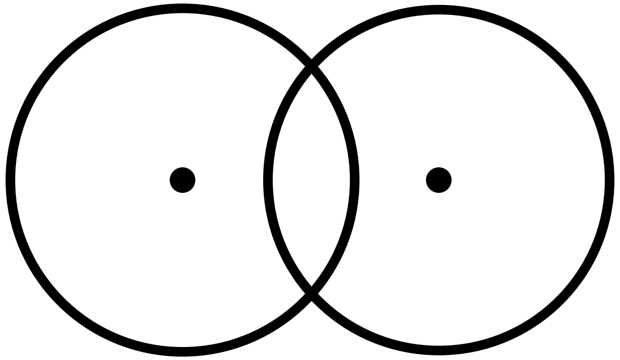

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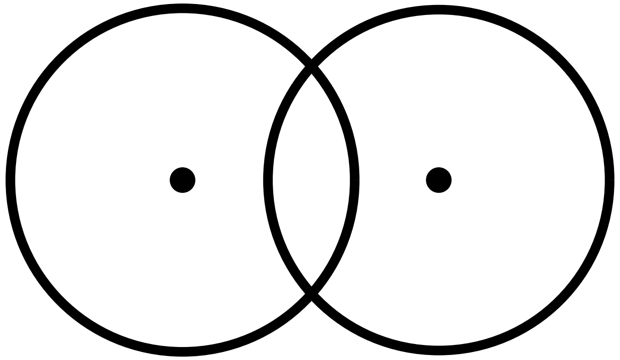
Then exists $t \in \mathcal{A}(L)$ and two points $p, q \in P$ for which

$d_{✂}(p, q) \leq d_{✂}(p, t) + d_{✂}(t, q) \leq 2r \leq \frac{4W}{\sqrt{n}} + 3$



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Hence $\#_{\times}(\ell) = O(\sqrt{n})$

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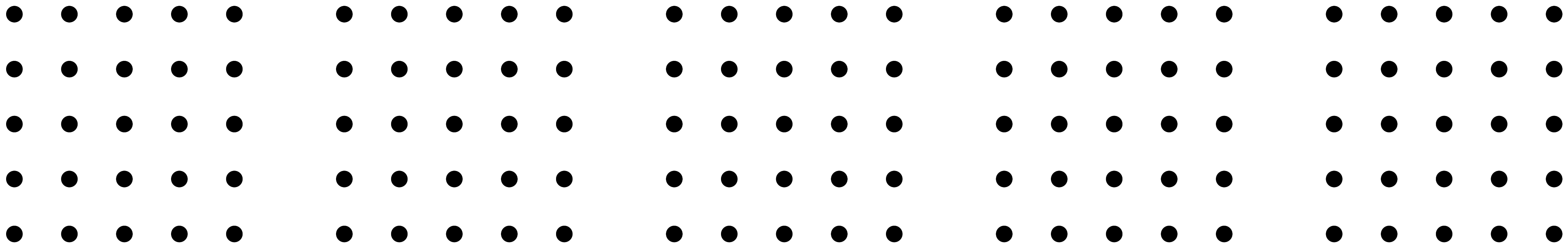

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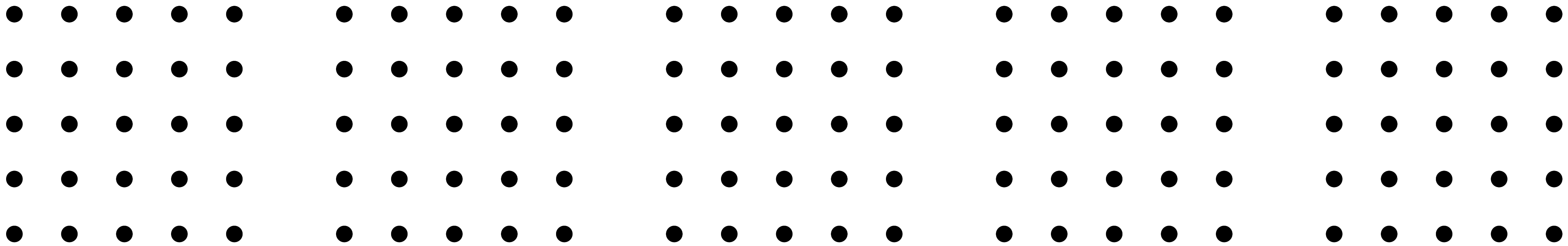

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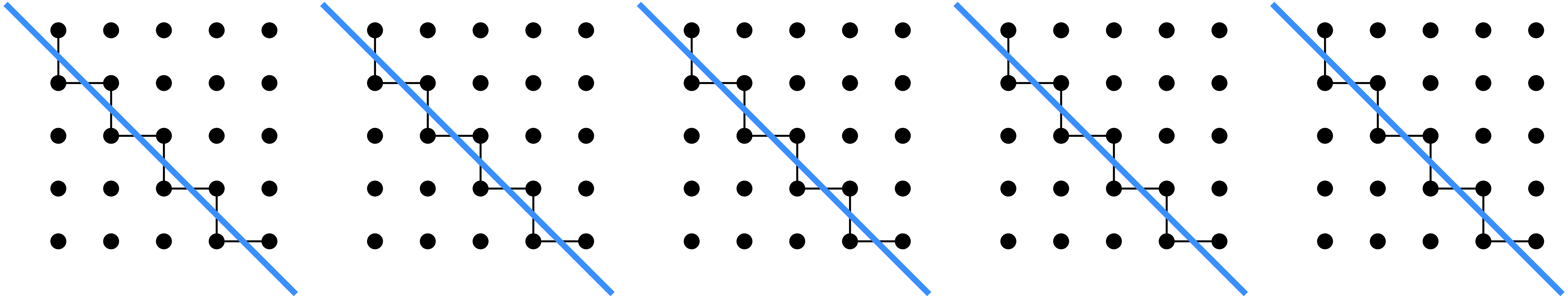

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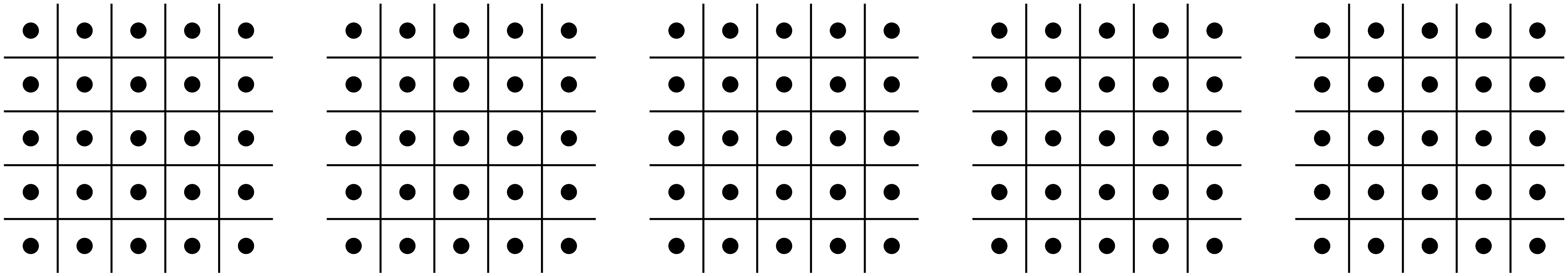

back to perfect matchings, discrepancy, and ε-samples

Perfect Matching with Low Stabbing Number

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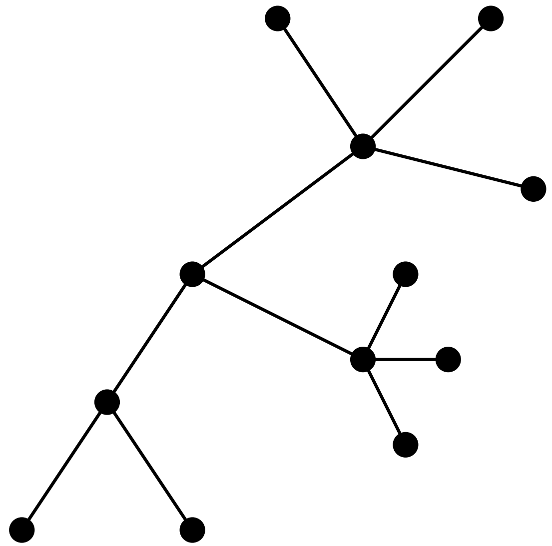

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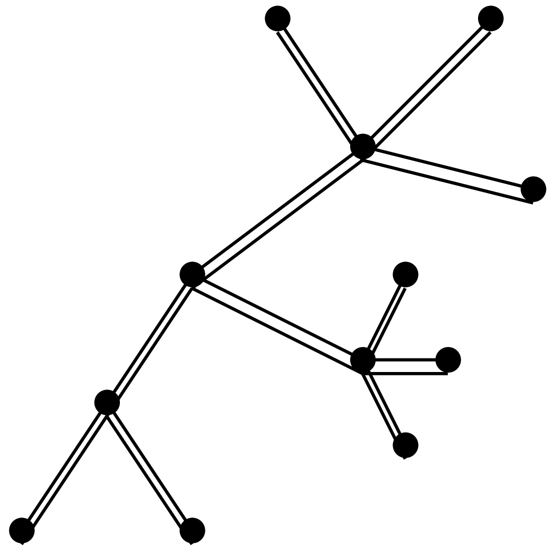
Stabbing number doubles



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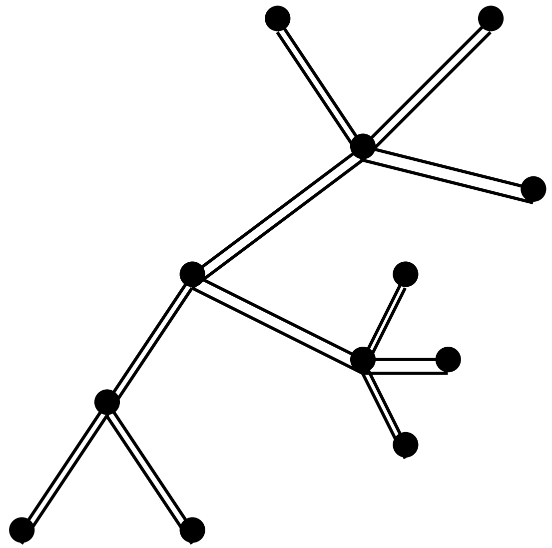

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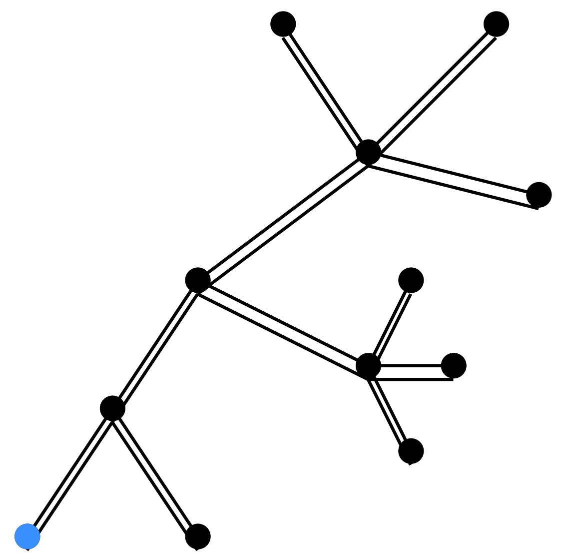

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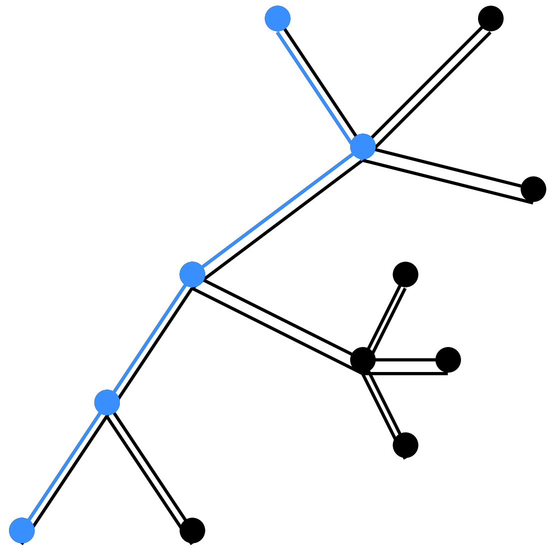

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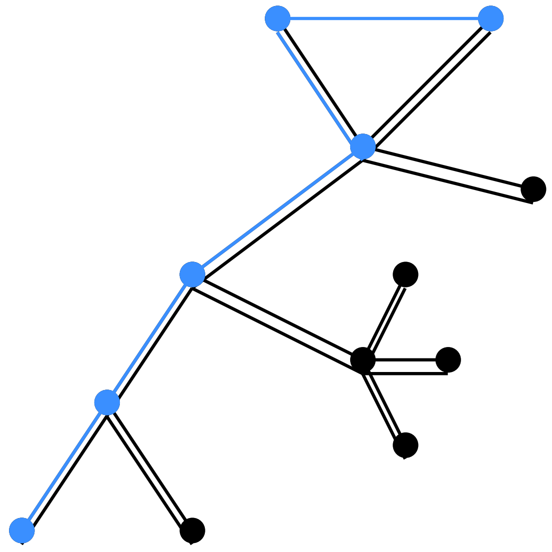

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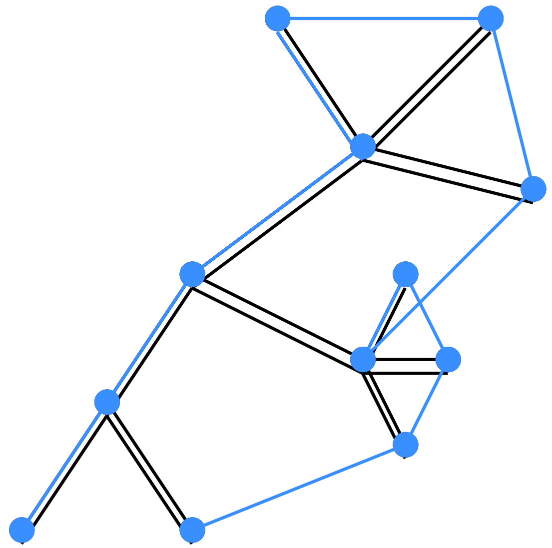

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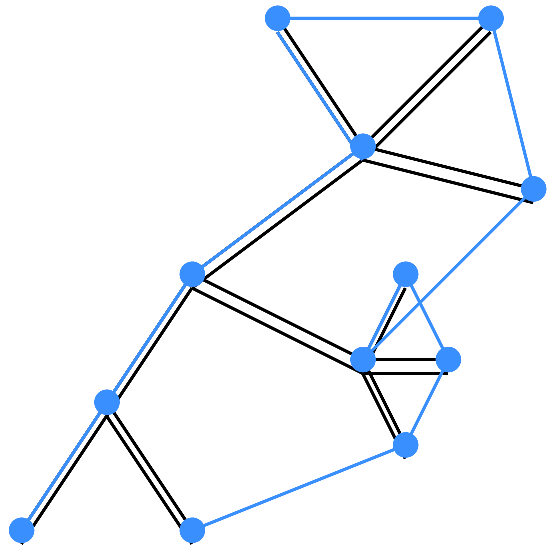

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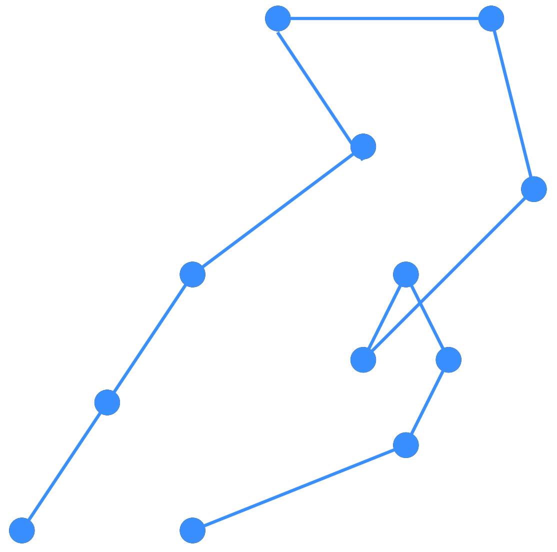

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$n - 1$ line segments in total

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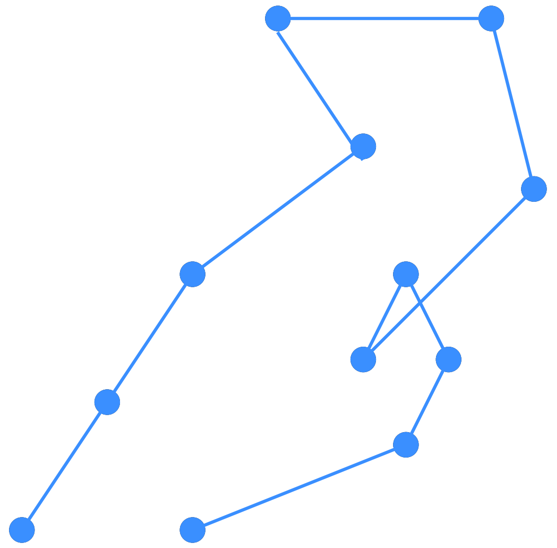

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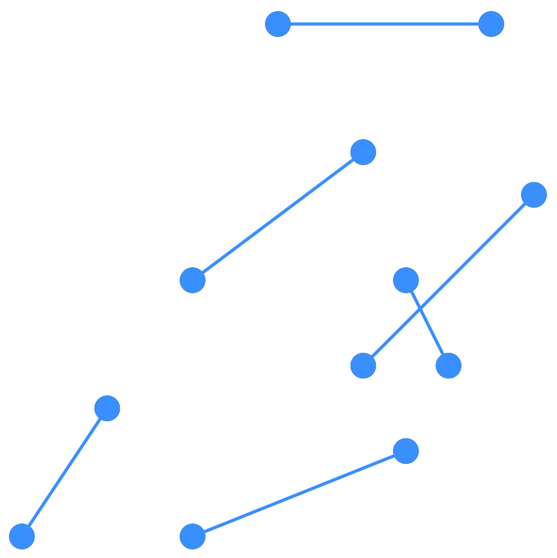
$n - 1$ line segments in total

If n even, remove all even edges to obtain perfect matching $\mathcal{M}$

Perfect Matching with Low Stabbing Number

Assume we have $\mathcal{T}$ with stabbing number $O(\sqrt{n})$



Double all line segments to obtain a Eulerian Graph
Stabbing number doubles

Then walk along the cycle

Skip any previously visited points

By triangle inequality, stabbing number can only decrease

$n-1$ line segments in total

If n even, remove all even edges to obtain perfect matching $\mathcal{M}$

Discrepancy and ε-sample

What we have now: For $\mathcal{R}$ = set of halfspaces, we get for $r \in \mathcal{R}$ a perfect matching s.t.

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Theorem. Given a range space $(X, \mathcal{R})$ with shattering dimension δ and dual shattering dimension δ^*, a set $P \subset X$ of n points, and $\varepsilon > 0$, one can compute, in polynomial time (assuming δ and δ^* are constant), an ε-sample for P of size

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This is smaller than our previous $O(1/\varepsilon^2)$!

Summary

We have seen

discrepancy (and there would be so much more too be said about discrepancy)

ε-samples via discrepancy (and we didn't even discuss how to use this for deterministic construction and/or ε-nets)

low-discrepancy colorings via perfect matchings (with low crossing number)

spanning trees with low crossing number (and therefore perfect matchings)

second application of reweighing

This was the last lecture about sampling.