Coresets (for directional width)

Geometric Approximation Algorithms

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setting: Geometric optimization problem on a point set P

coreset: Small subset $S \subset P$ that captures the structure of optimal solution

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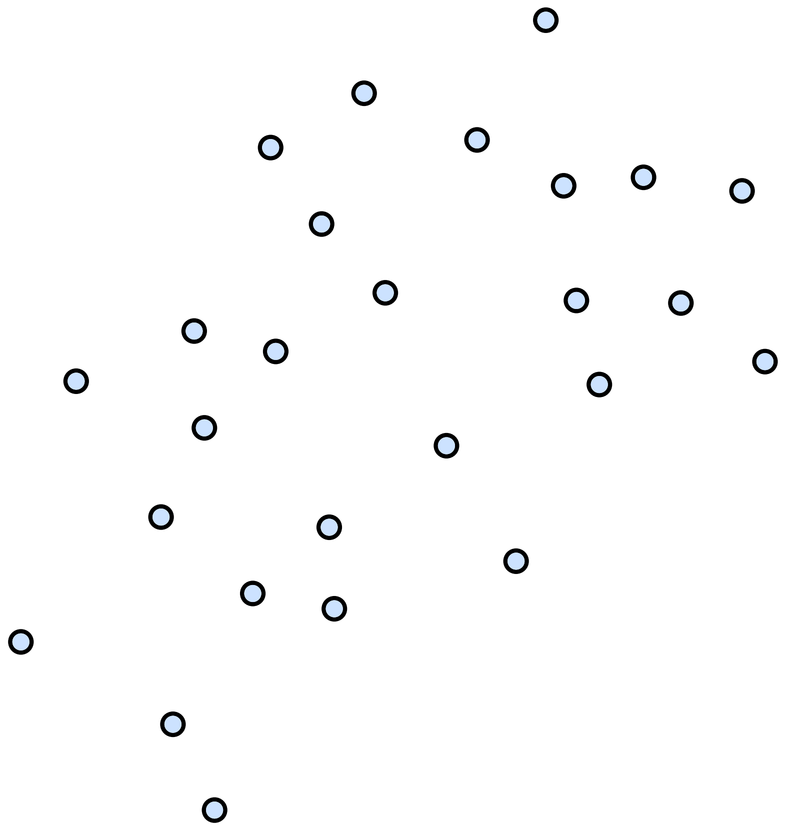

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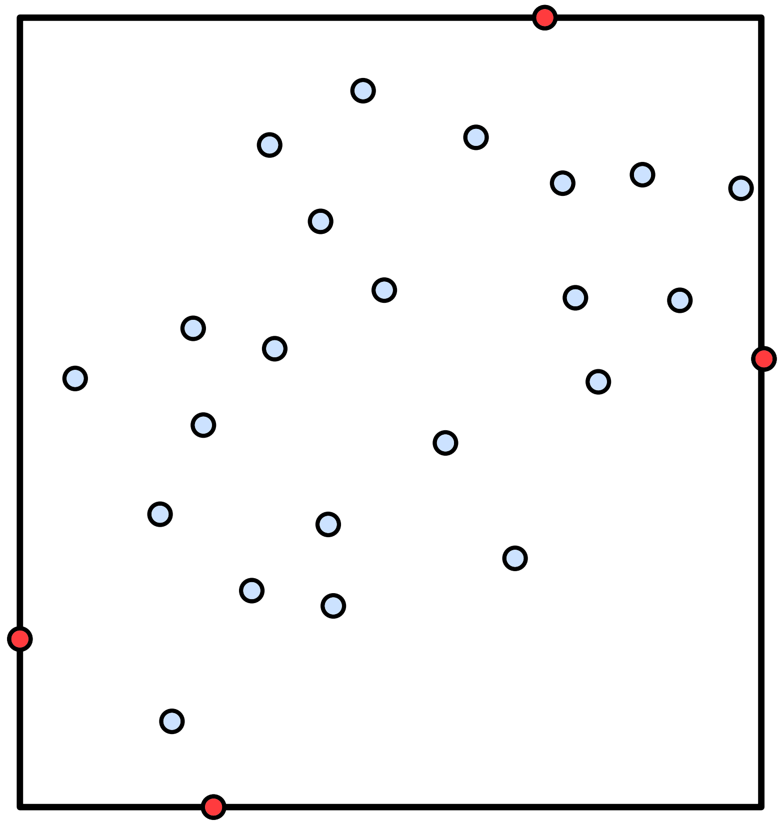
S : points with minimum and maximum coordinate (for each coordinate)



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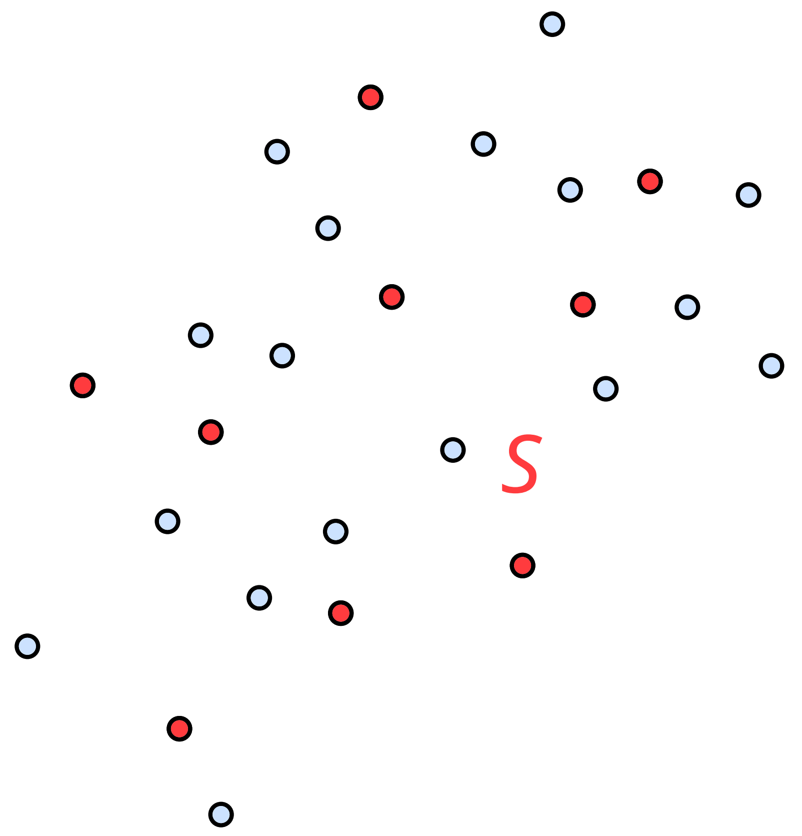

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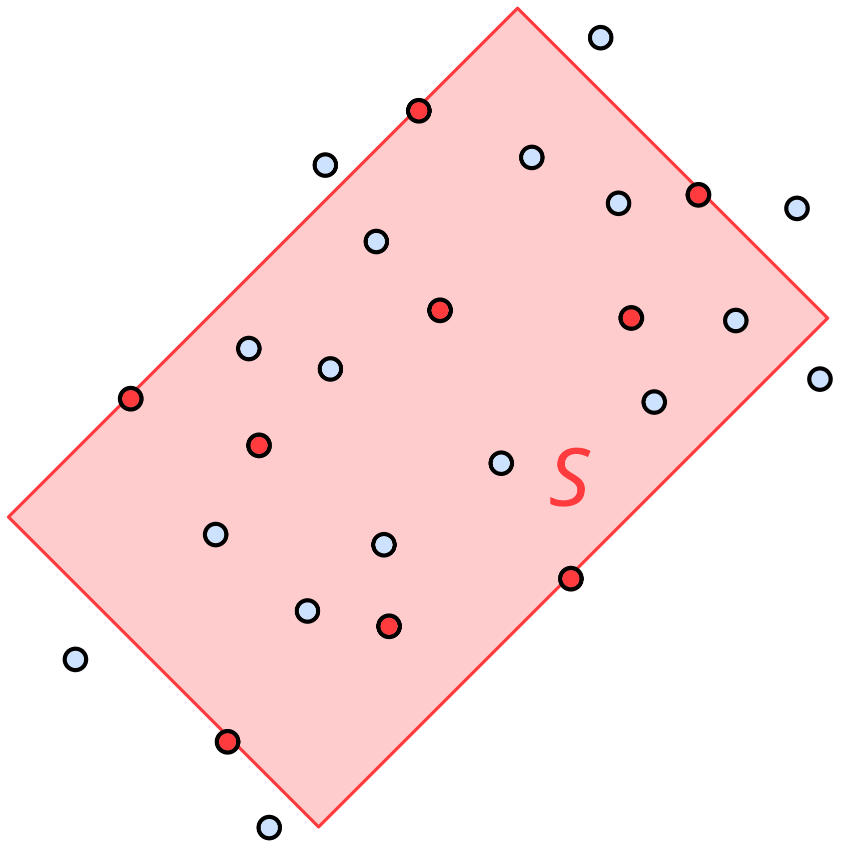

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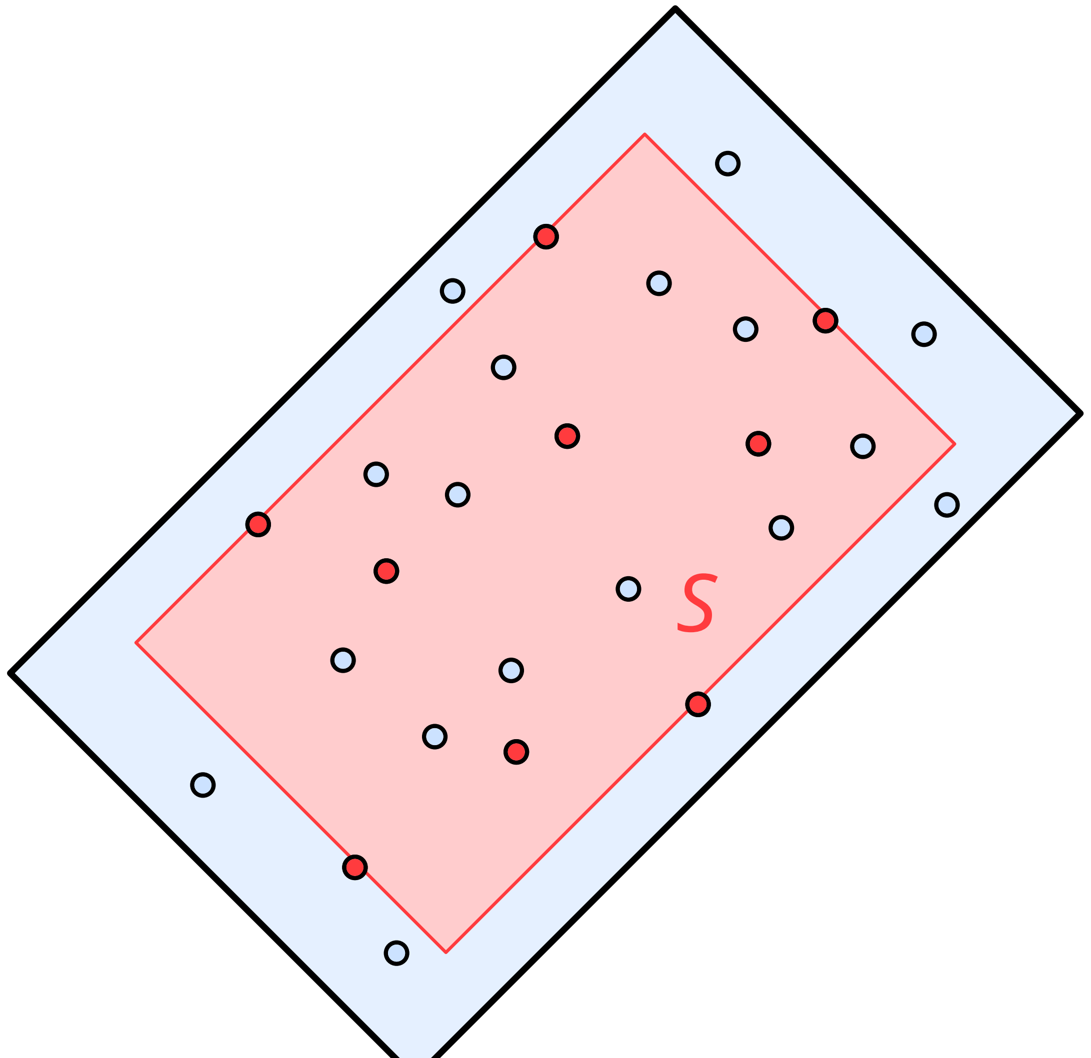

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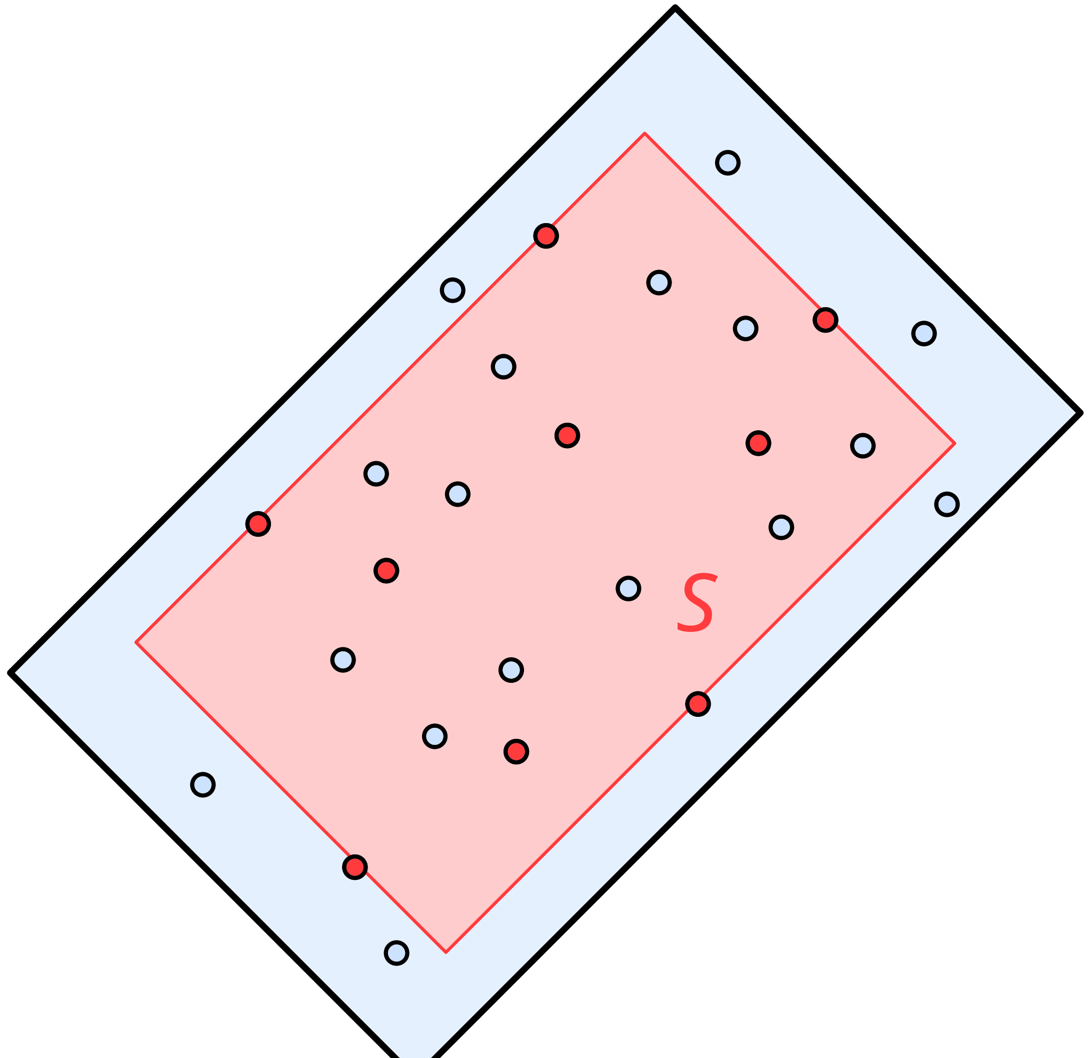

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combined: $(1 + \varepsilon)$-approximation in $O(n + 1/\varepsilon^{9/2})$ time

Overview

Coreset for directional width

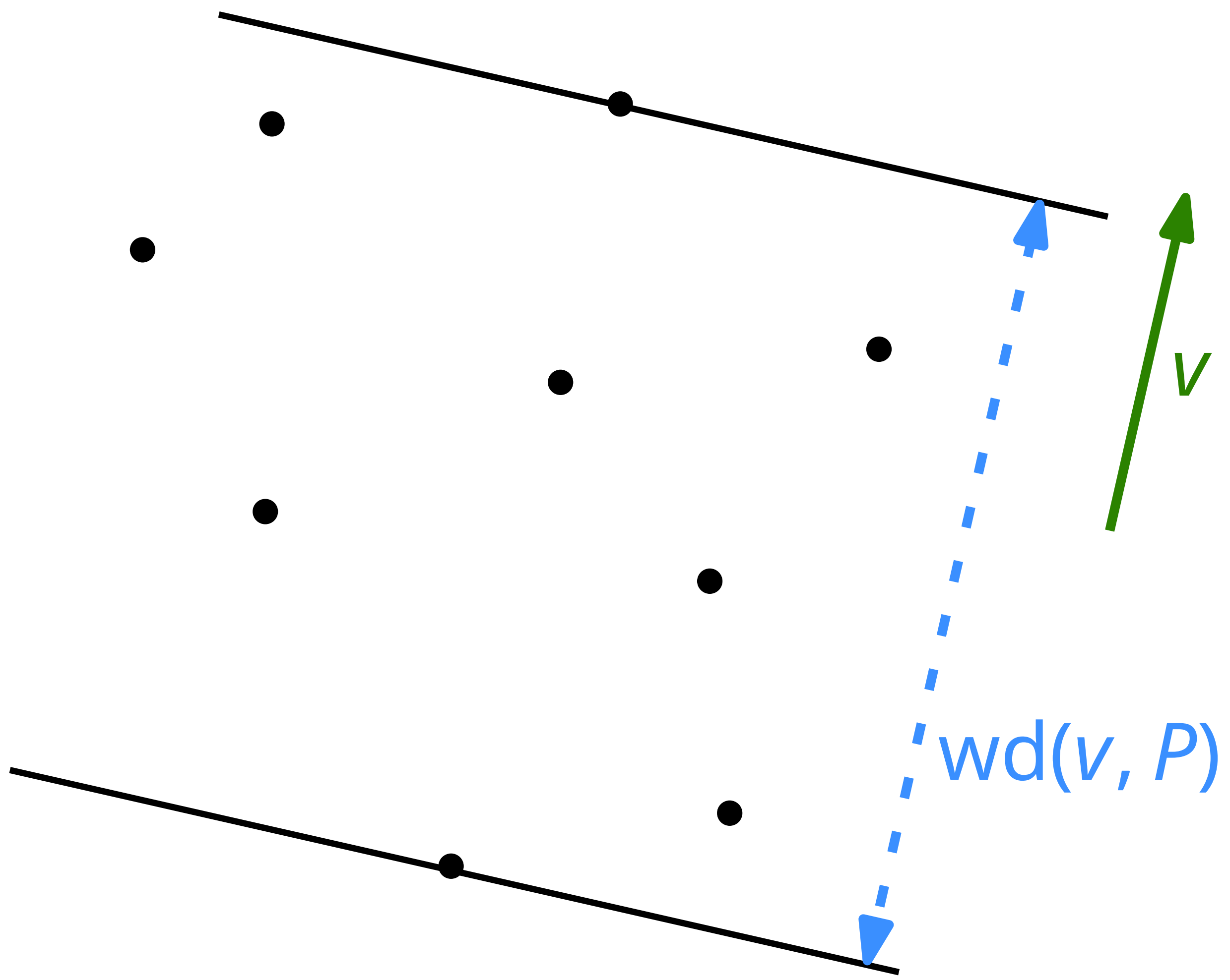
- definition
- applications
- construction algorithm

Extra ingredient: Minimum volume bounding box

Directional Width

Definition: The directional width of $P \subset \mathbb{R}^d$ with respect to a vector $v \in \mathbb{R}^d \setminus \{0\}$ is

$$\mathrm{wd}(v, P) := \max_{p \in P} \langle v, p \rangle - \min_{p \in P} \langle v, p \rangle$$



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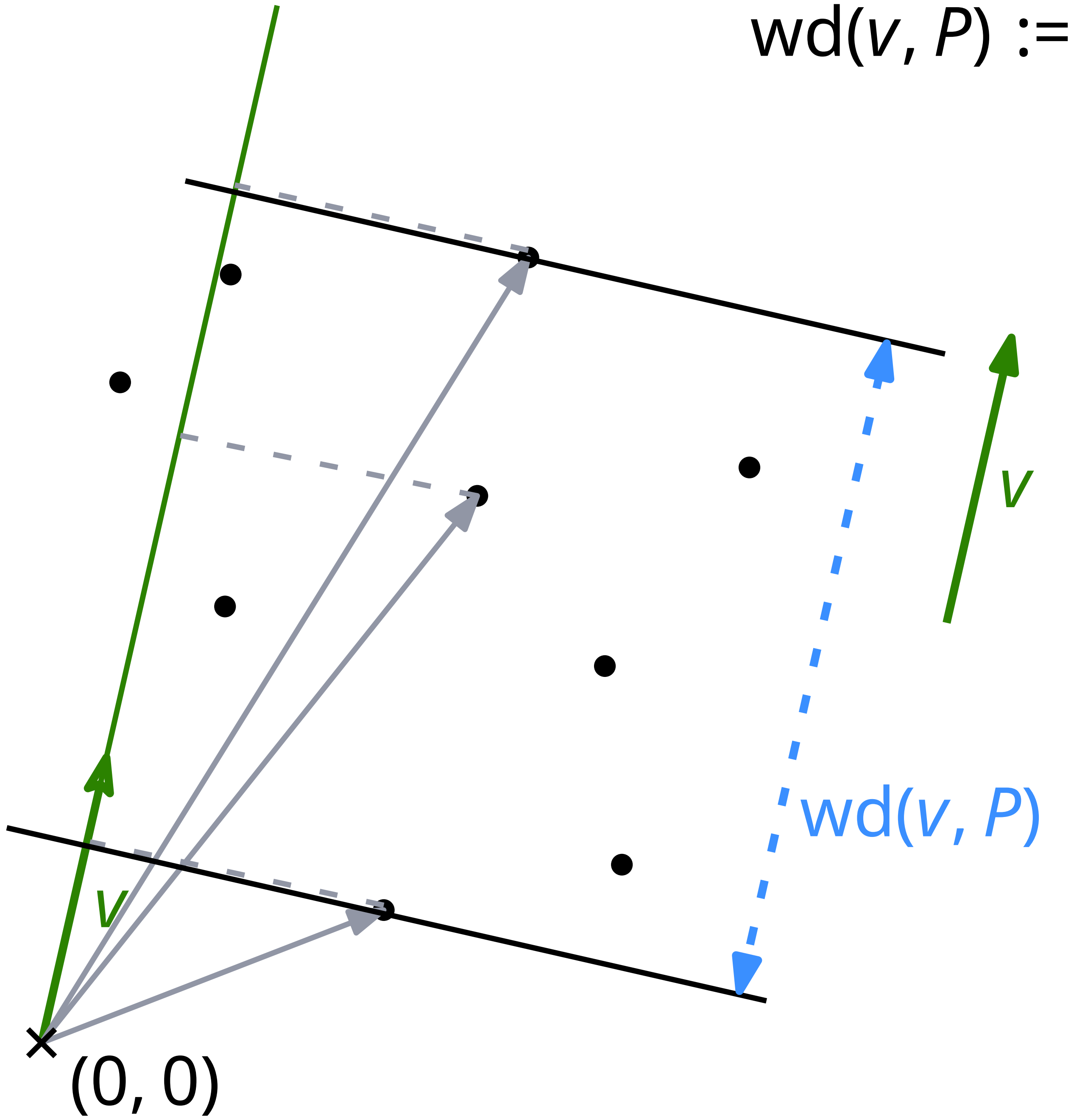

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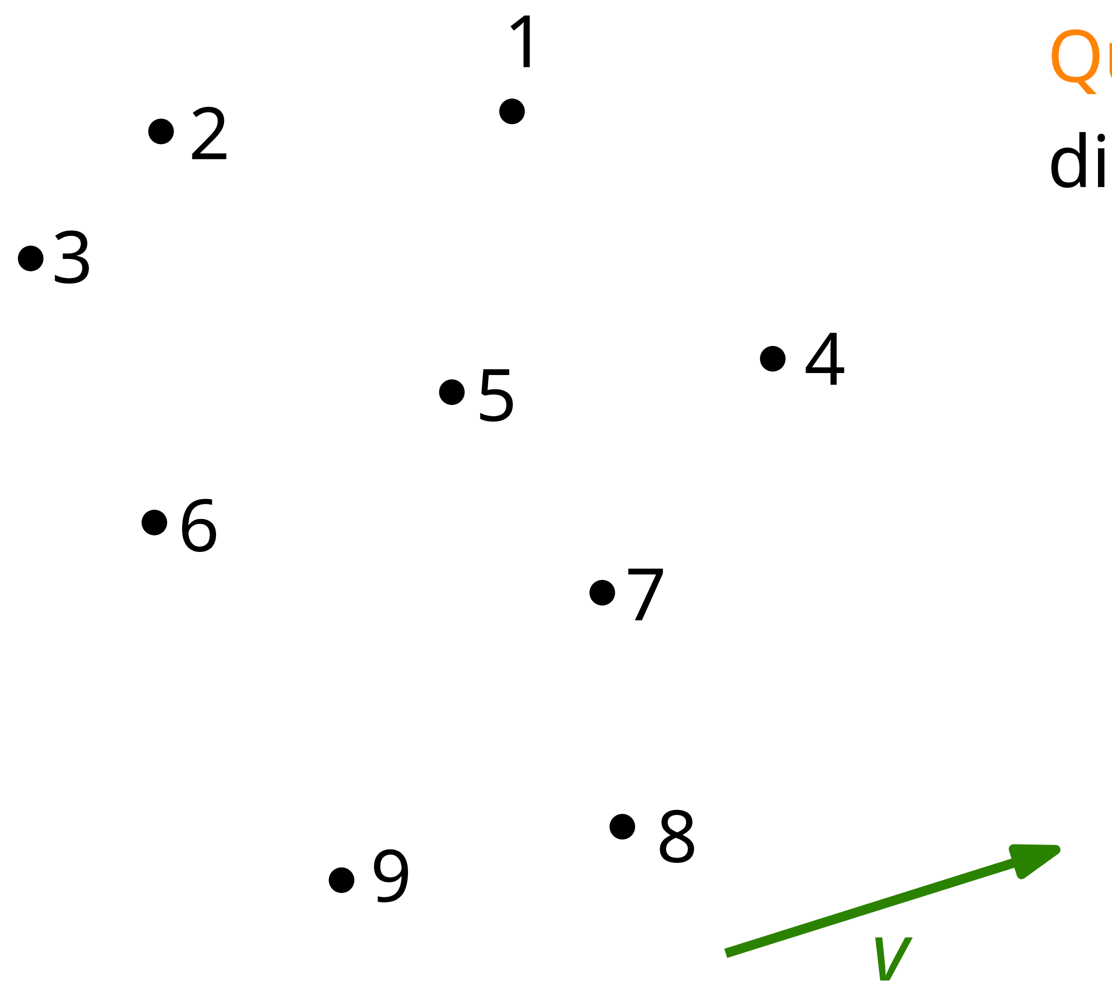

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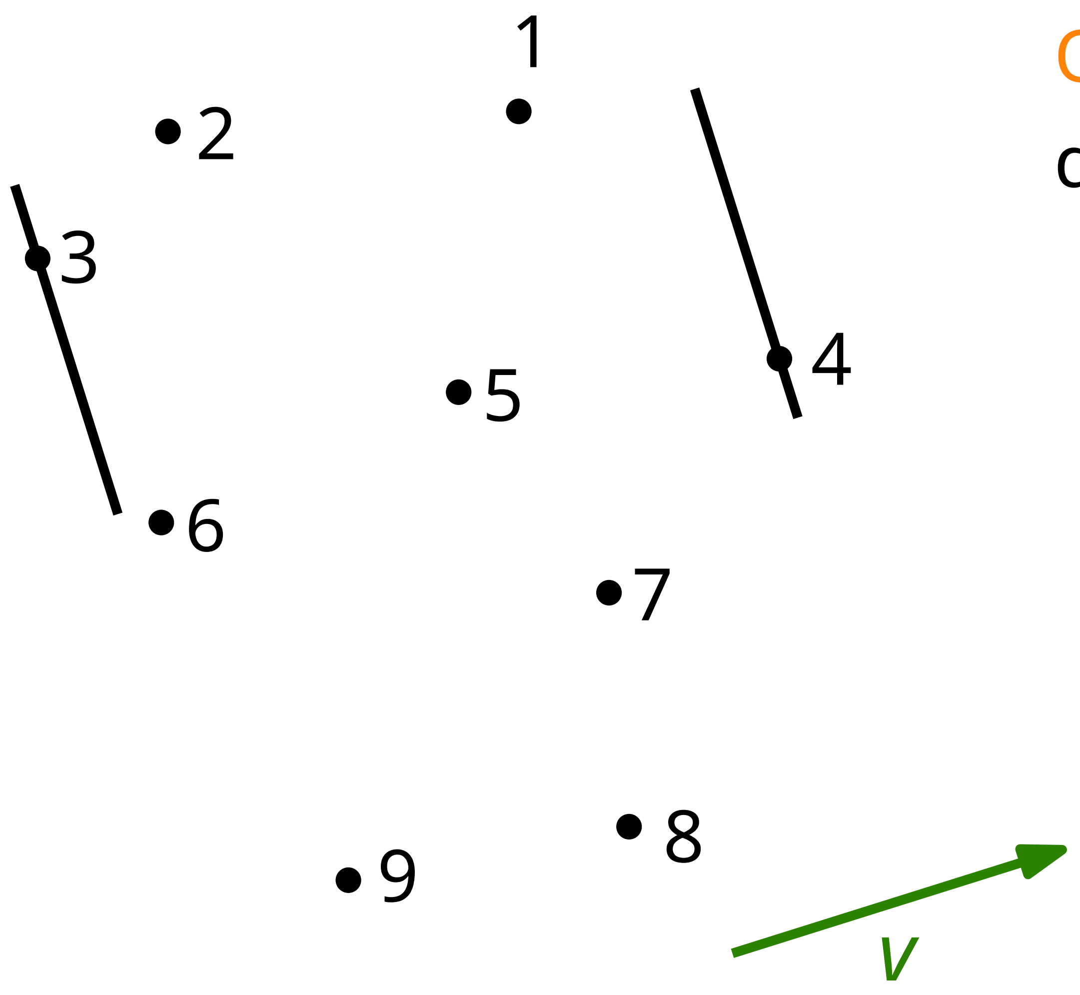

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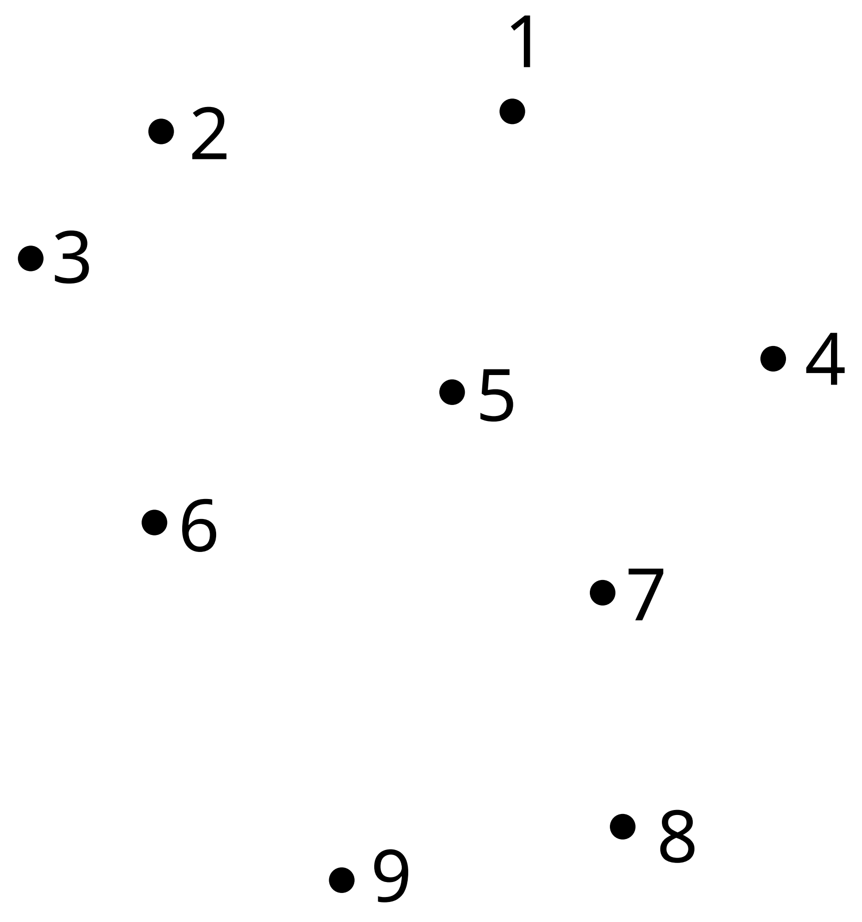

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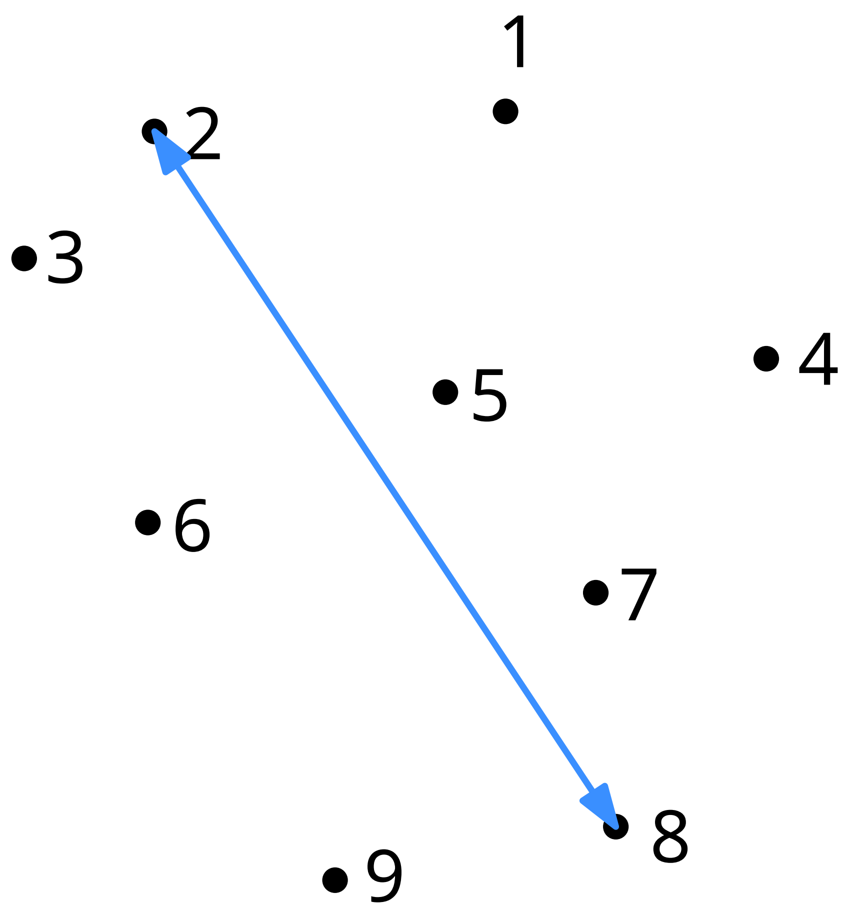

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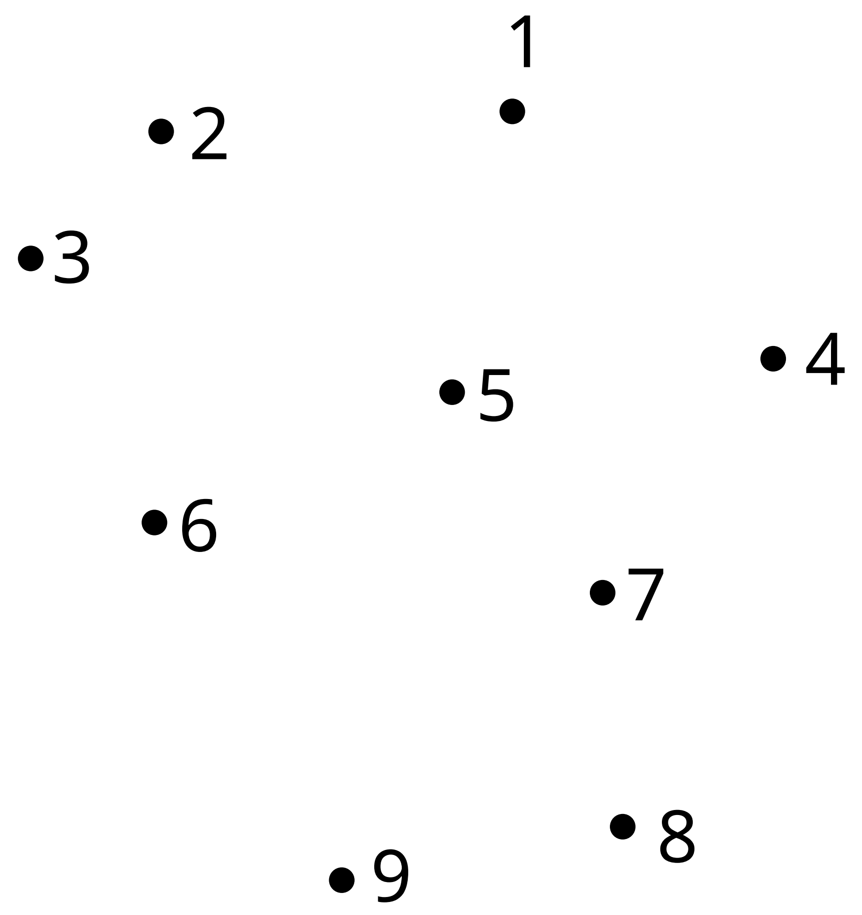

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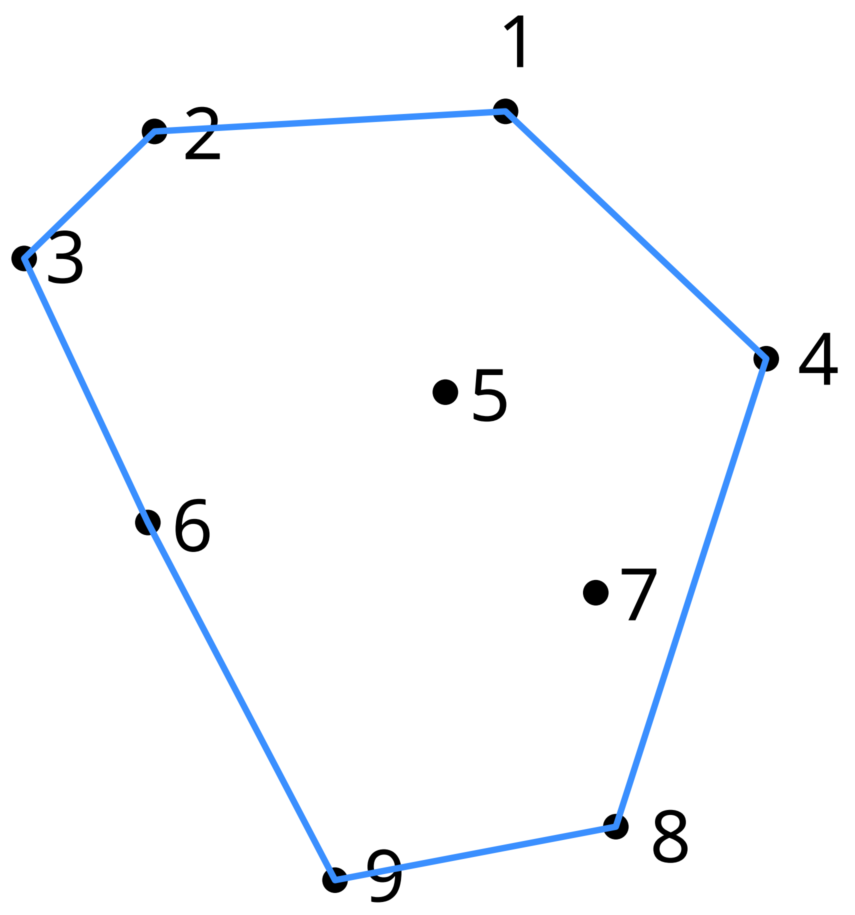

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Only points on convex hull
(i.e. 1, 2, 3, 6, 9, 8, 4)

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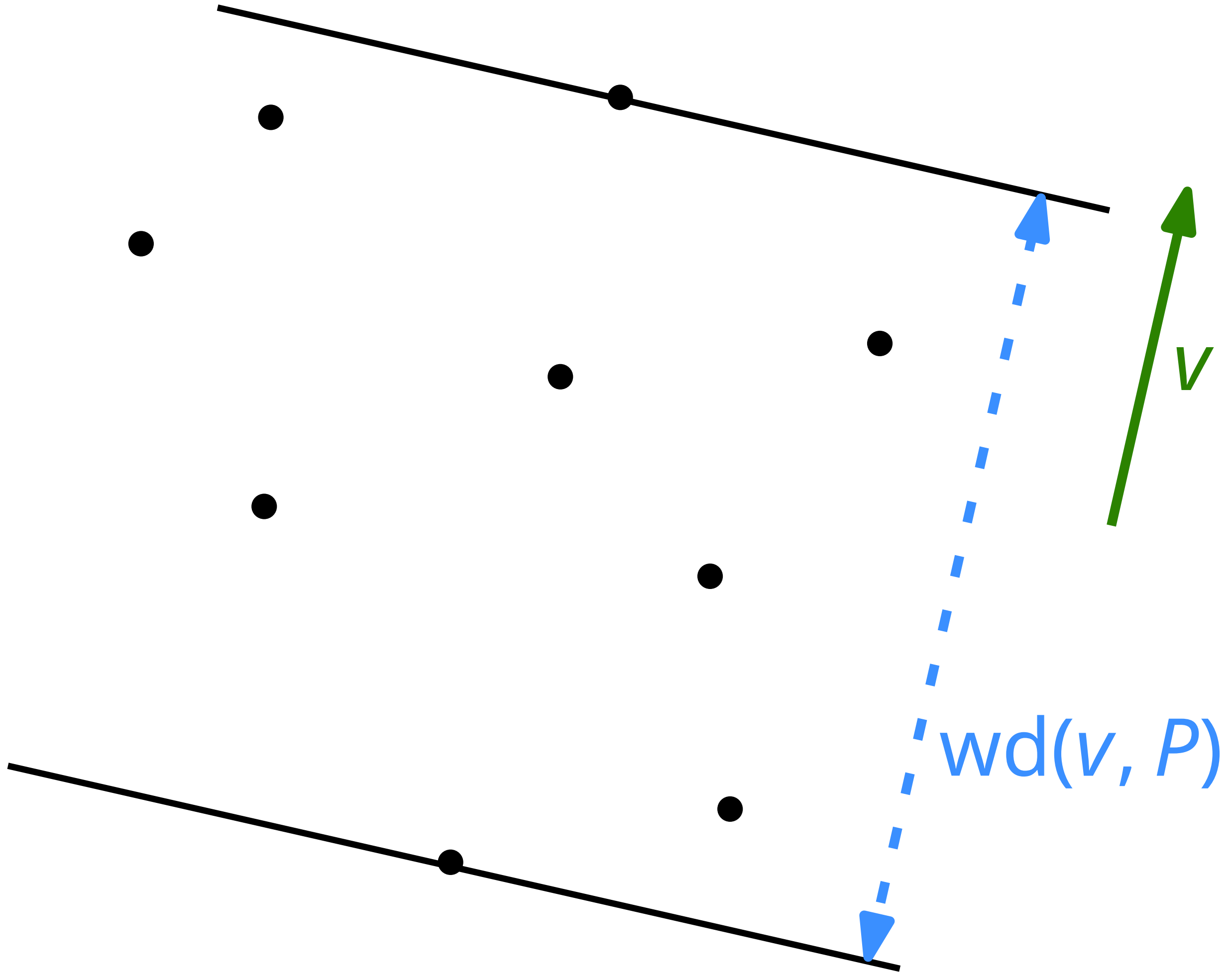

Properties:

- translation invariant
- scales linearly
- $\mathrm{wd}(v, P) = \mathrm{wd}(v, \mathrm{conv}(P))$
- monotone: if $Q \subset P$, then $\mathrm{wd}(v, Q) \leq \mathrm{wd}(v, P)$

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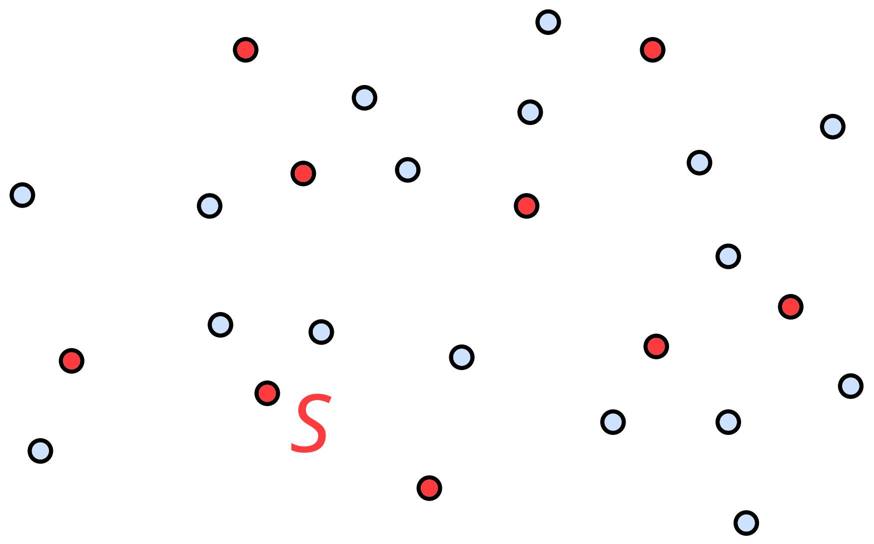

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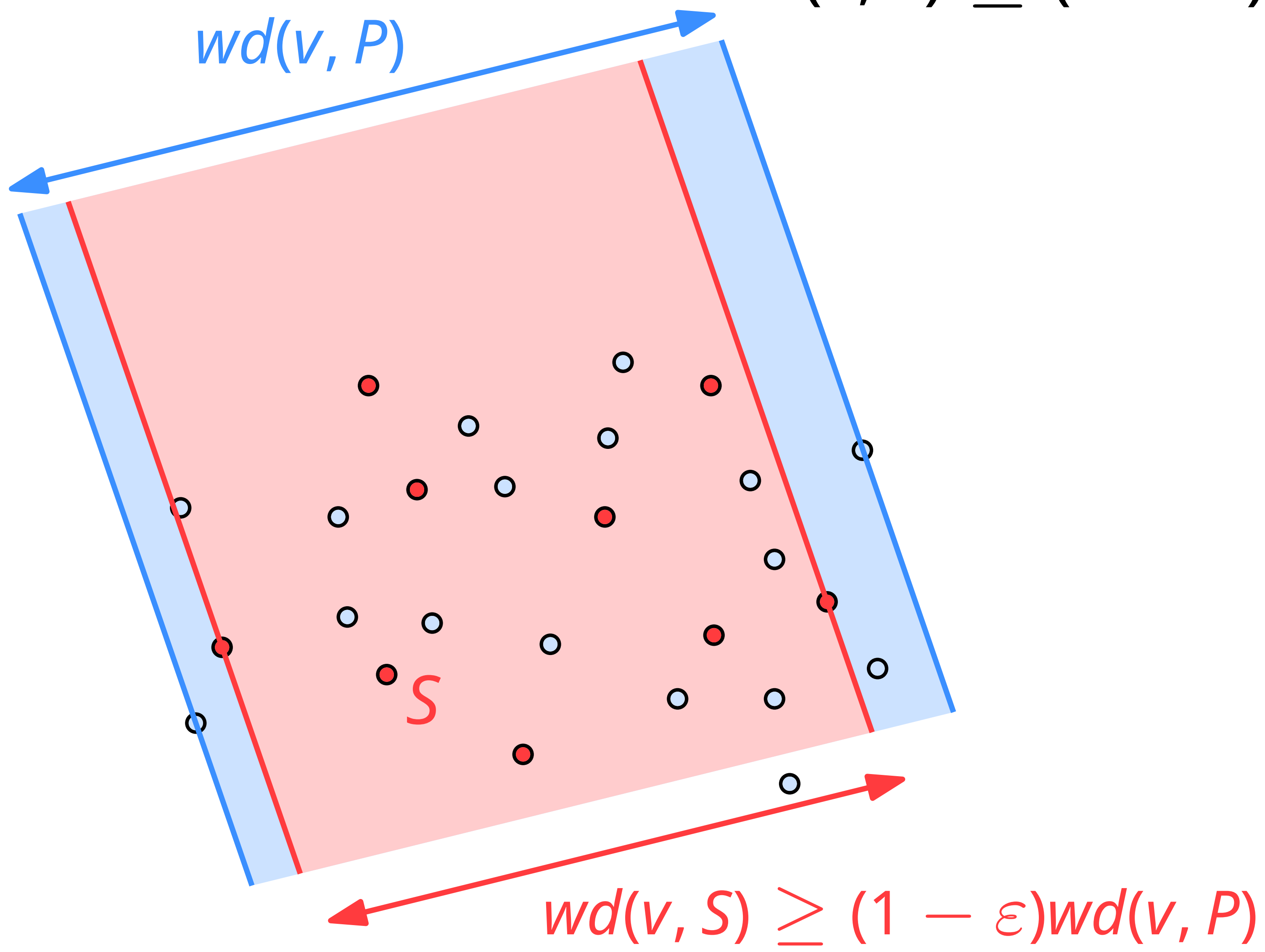

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- general concept: other objective functions f instead of directional width

Overview

Coreset for directional width

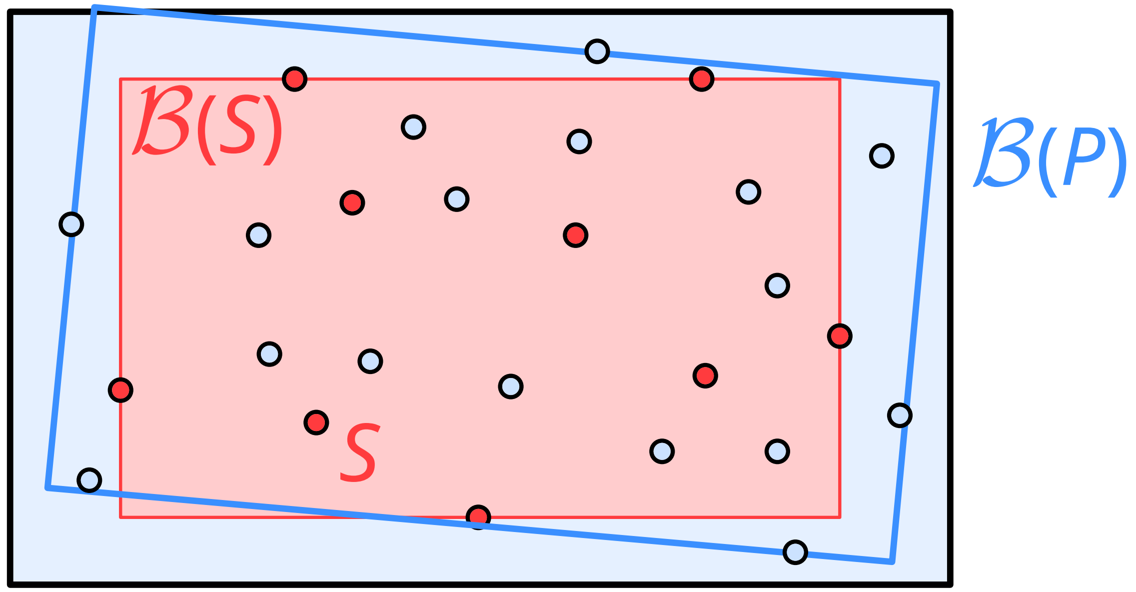
- definition
- applications ←
- construction algorithm

Extra ingredient: Minimum volume bounding box

Use case: min volume bounding box

Given $\varepsilon > 0$, $P \subset \mathbb{R}^d$, let S be a δ-coreset of P for directional width ($\delta = \varepsilon/(8d)$). Let $\mathcal{B}(P)$ (resp. $\mathcal{B}(S)$) be the min volume bounding box of P (resp. S), and let B be $\mathcal{B}(S)$ scaled by $(1 + 3\delta)$ around the center of $\mathcal{B}(S)$.

Then $P \subset B$, and $\mathrm{Vol}(B) \leq (1 + \varepsilon)\mathrm{Vol}(\mathcal{B}(P))$.



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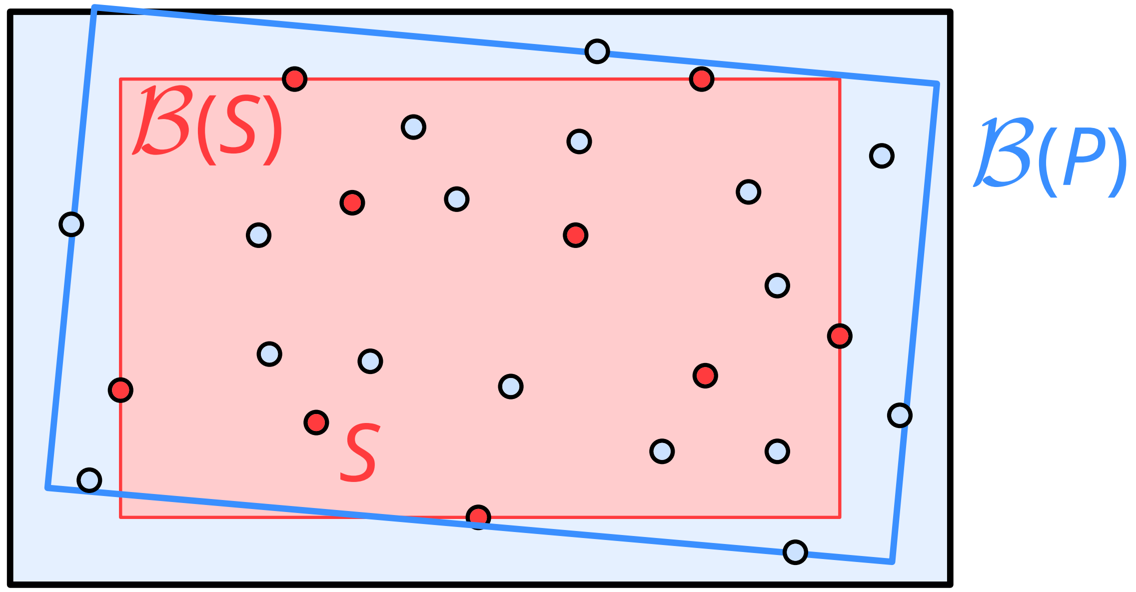

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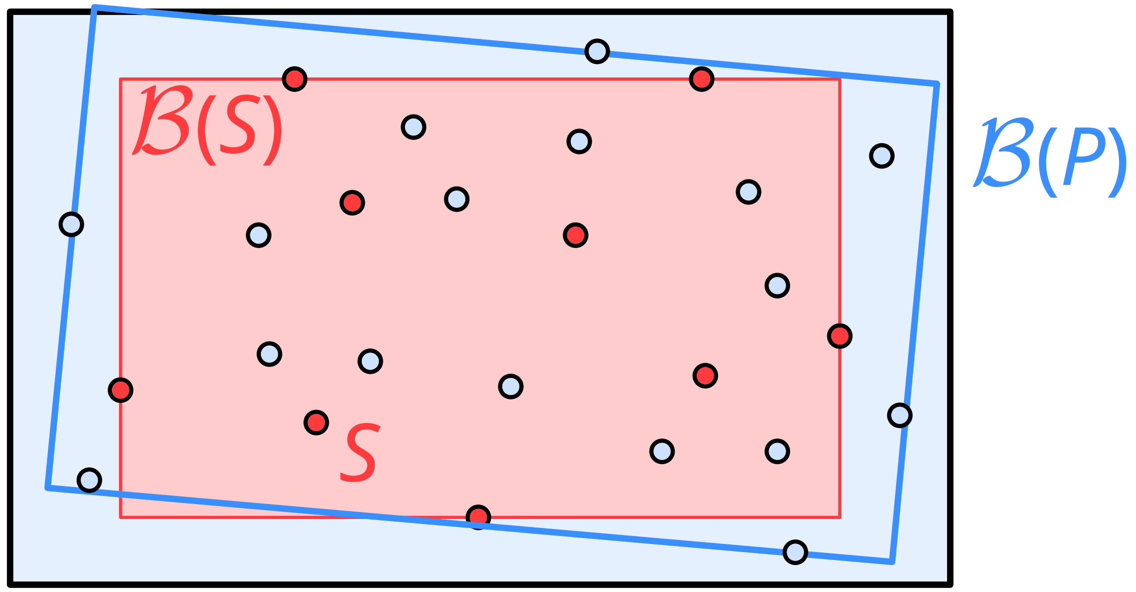

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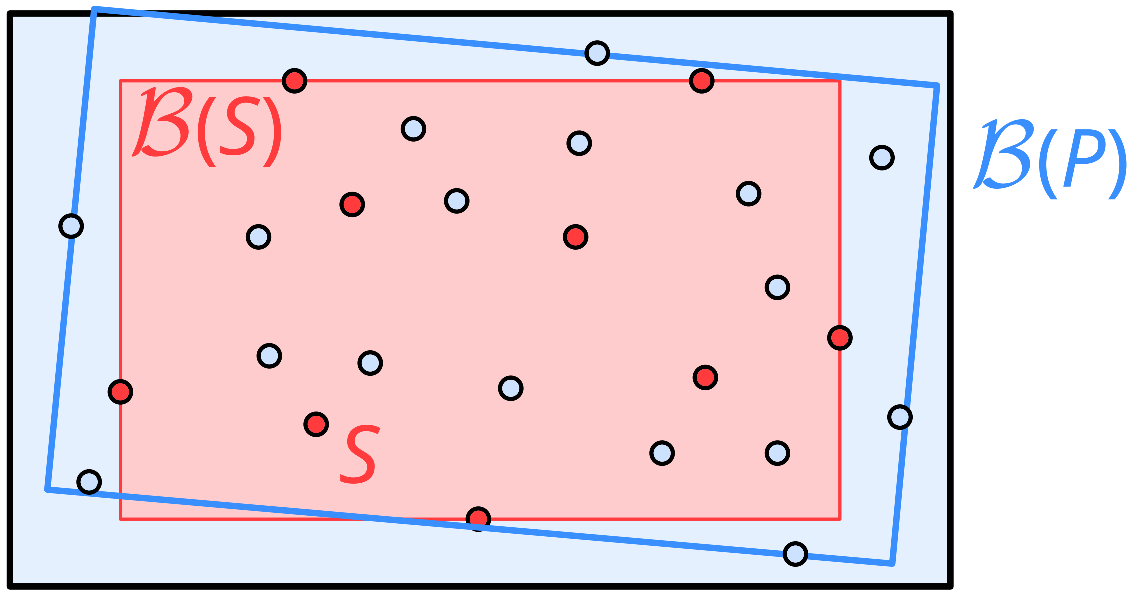

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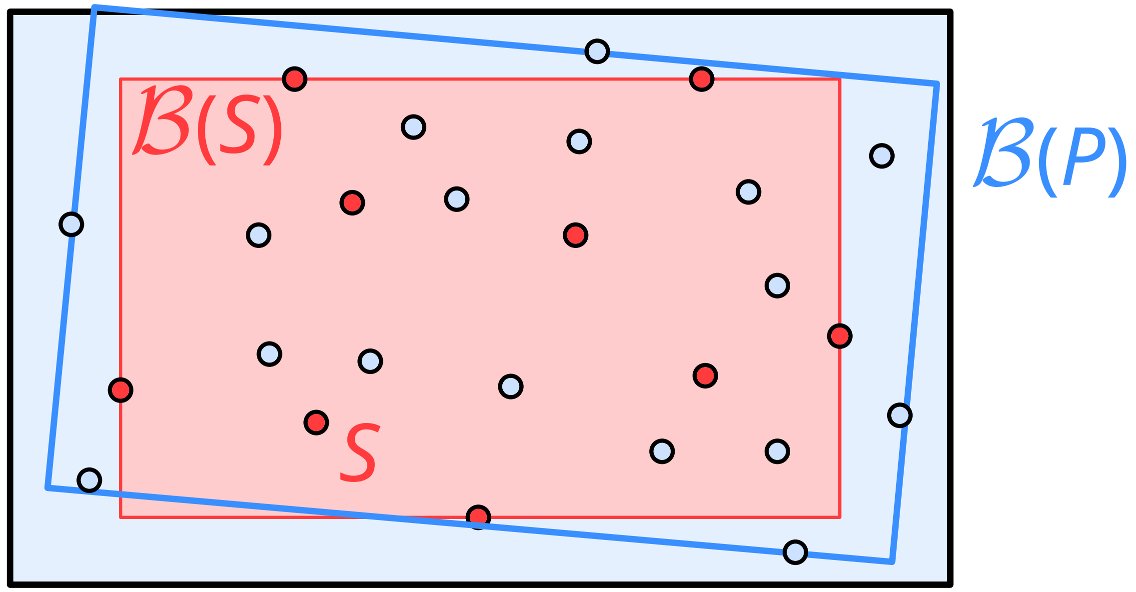

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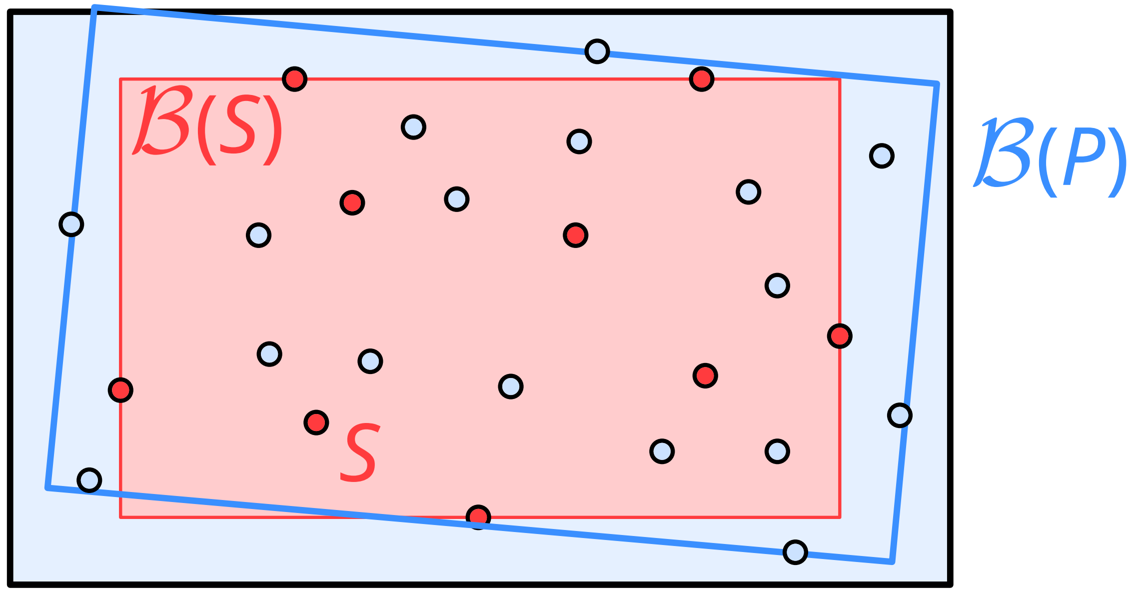

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Still need: B contains P

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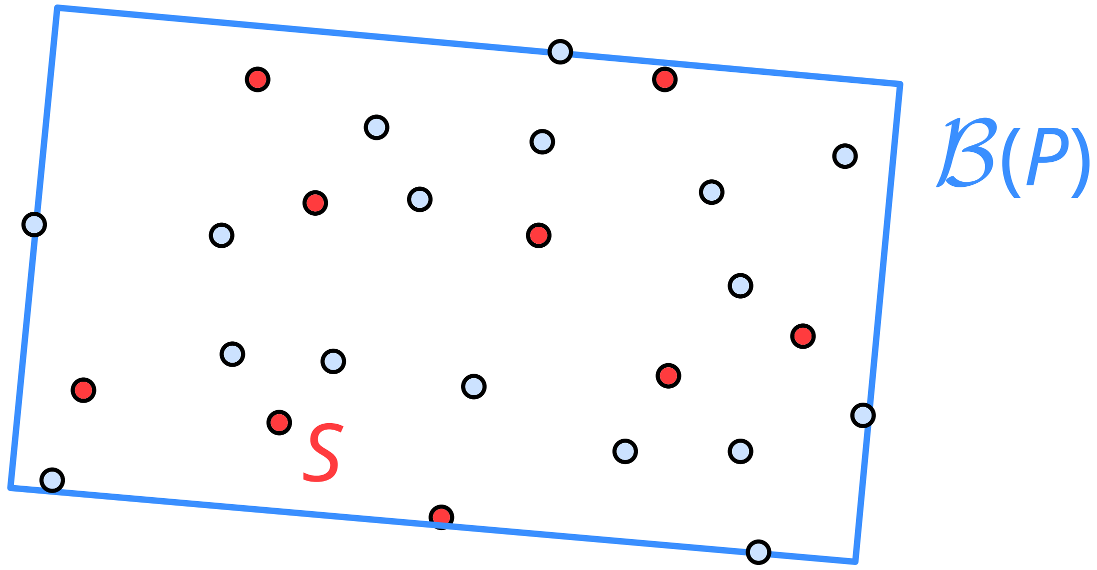

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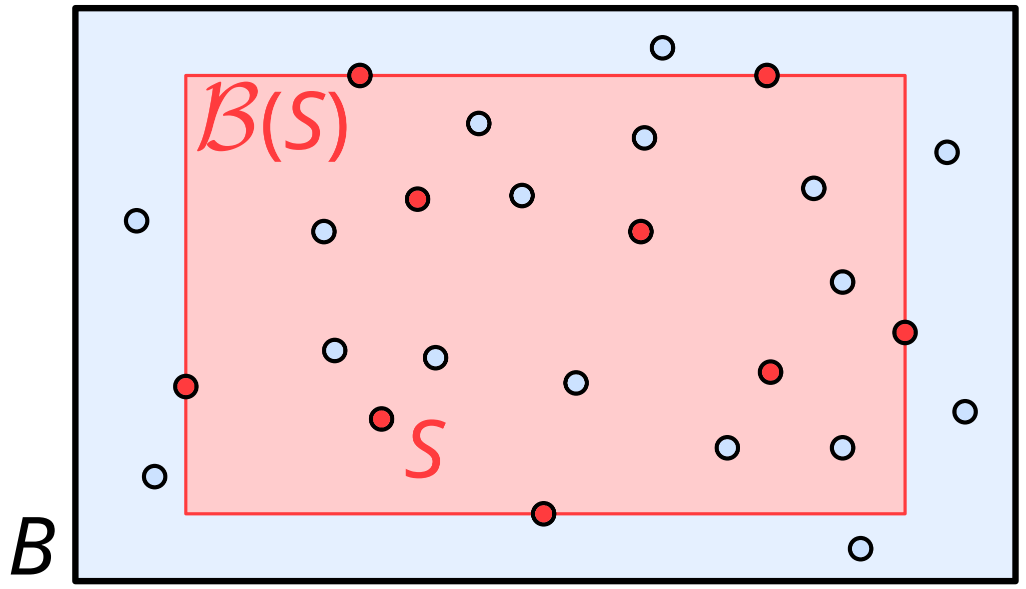

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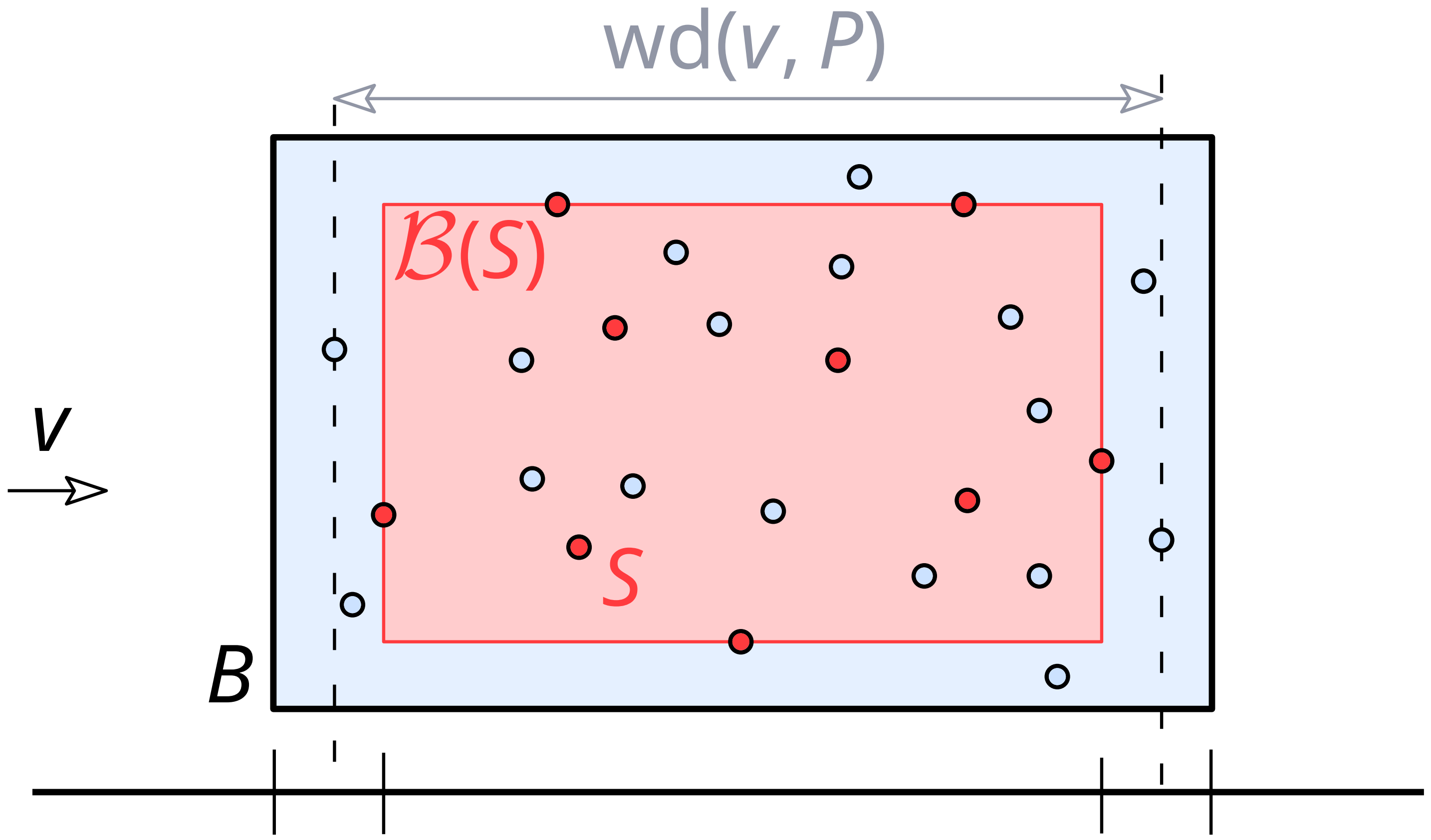

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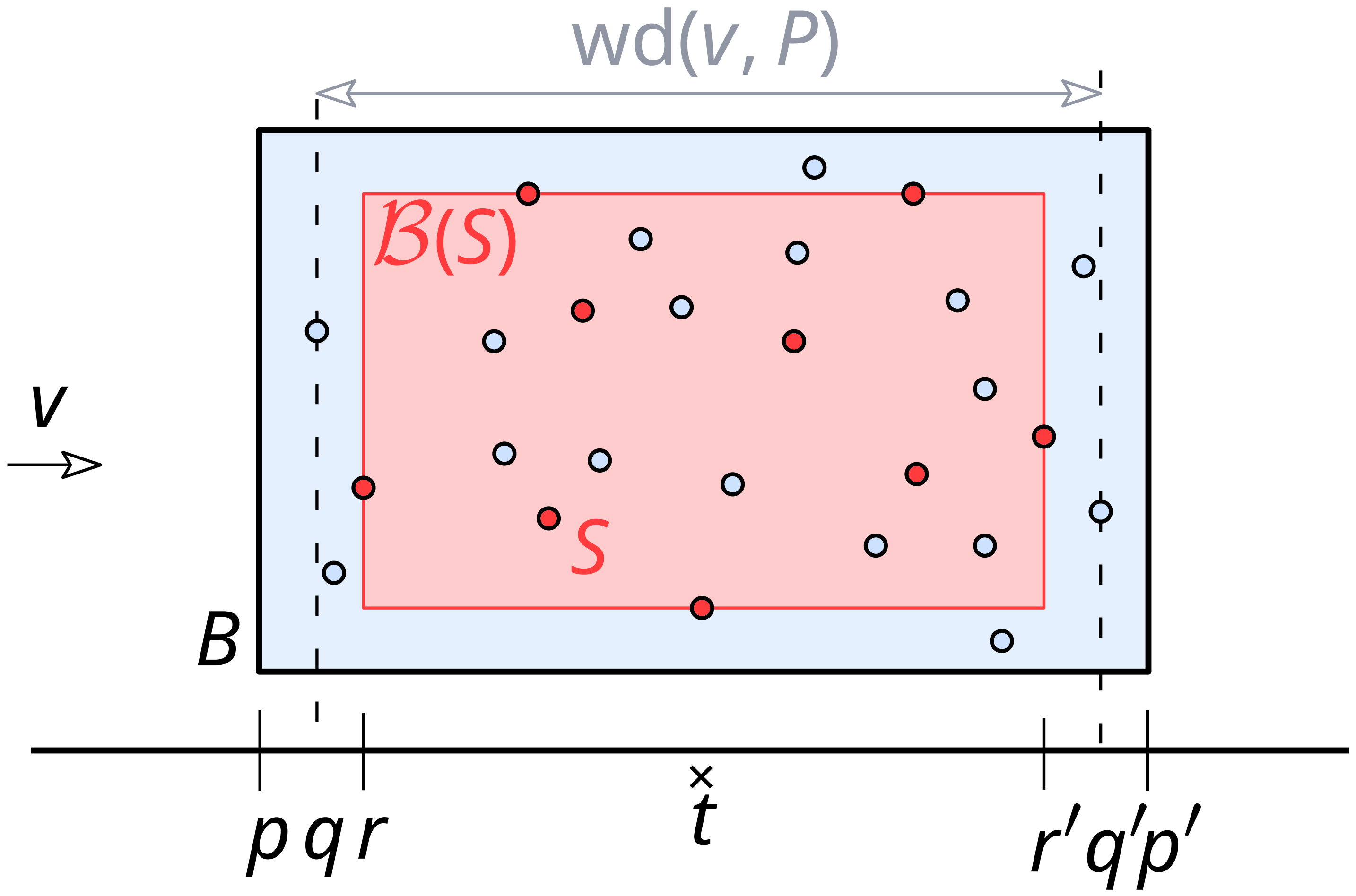

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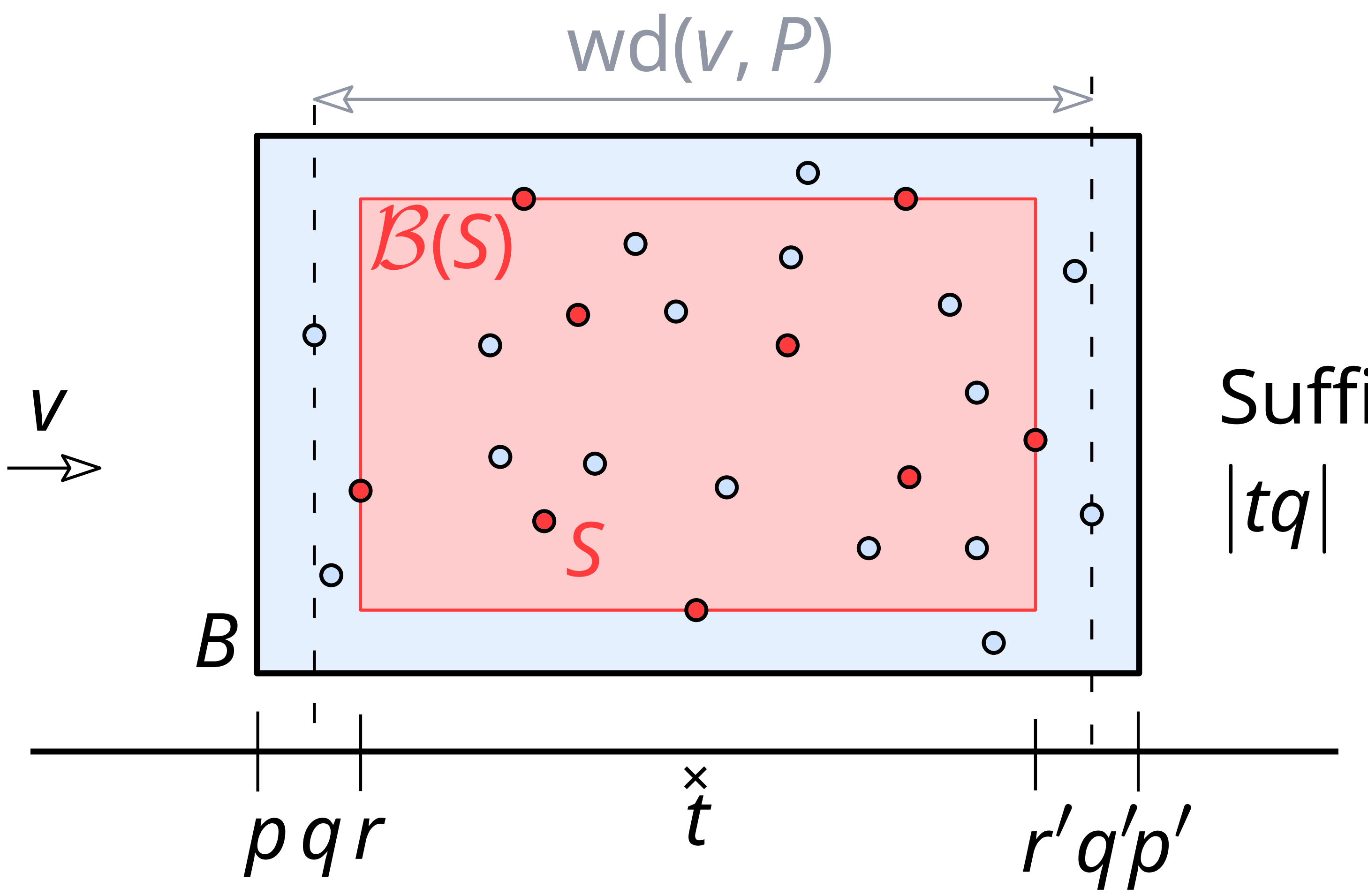

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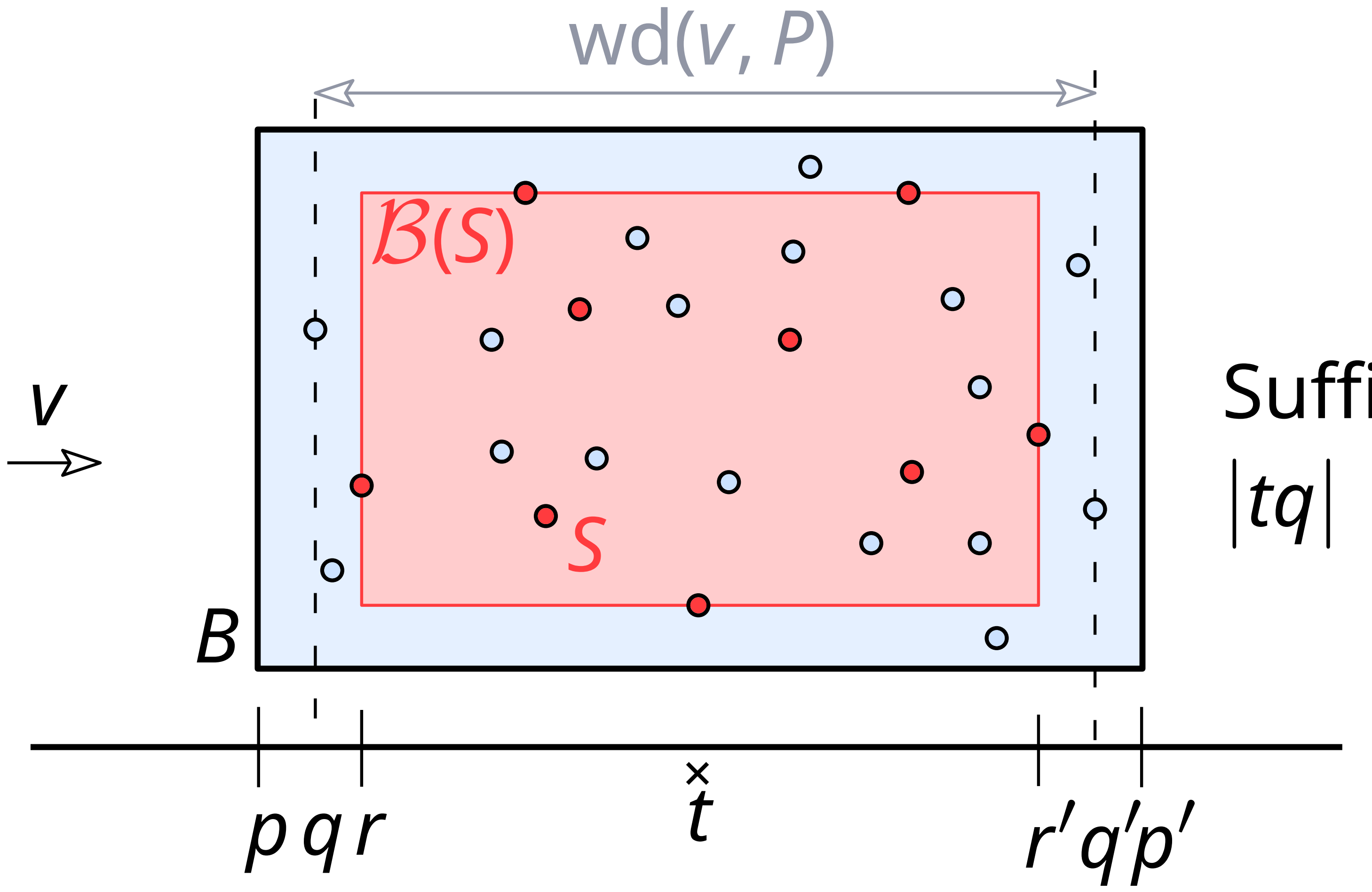

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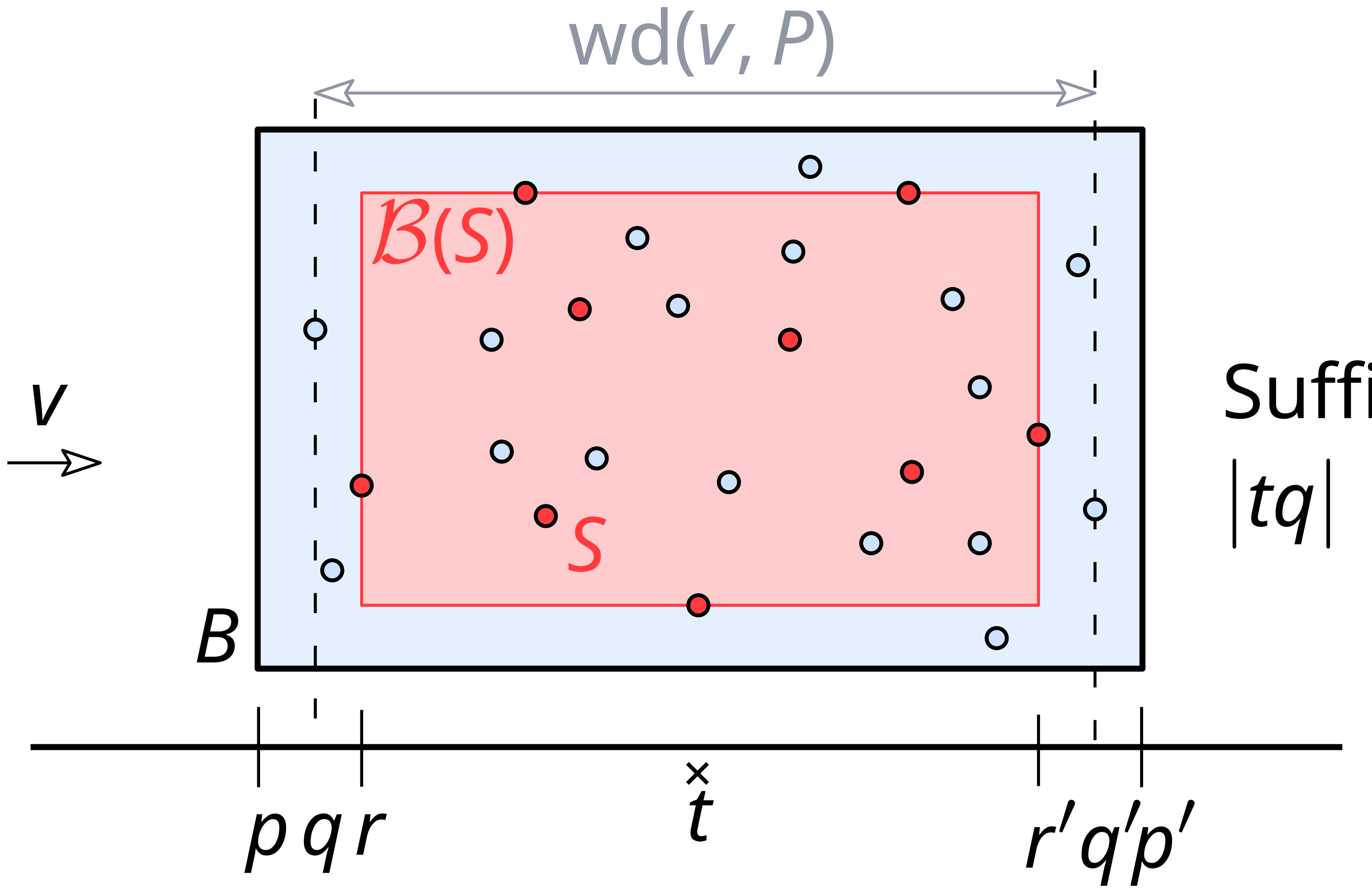

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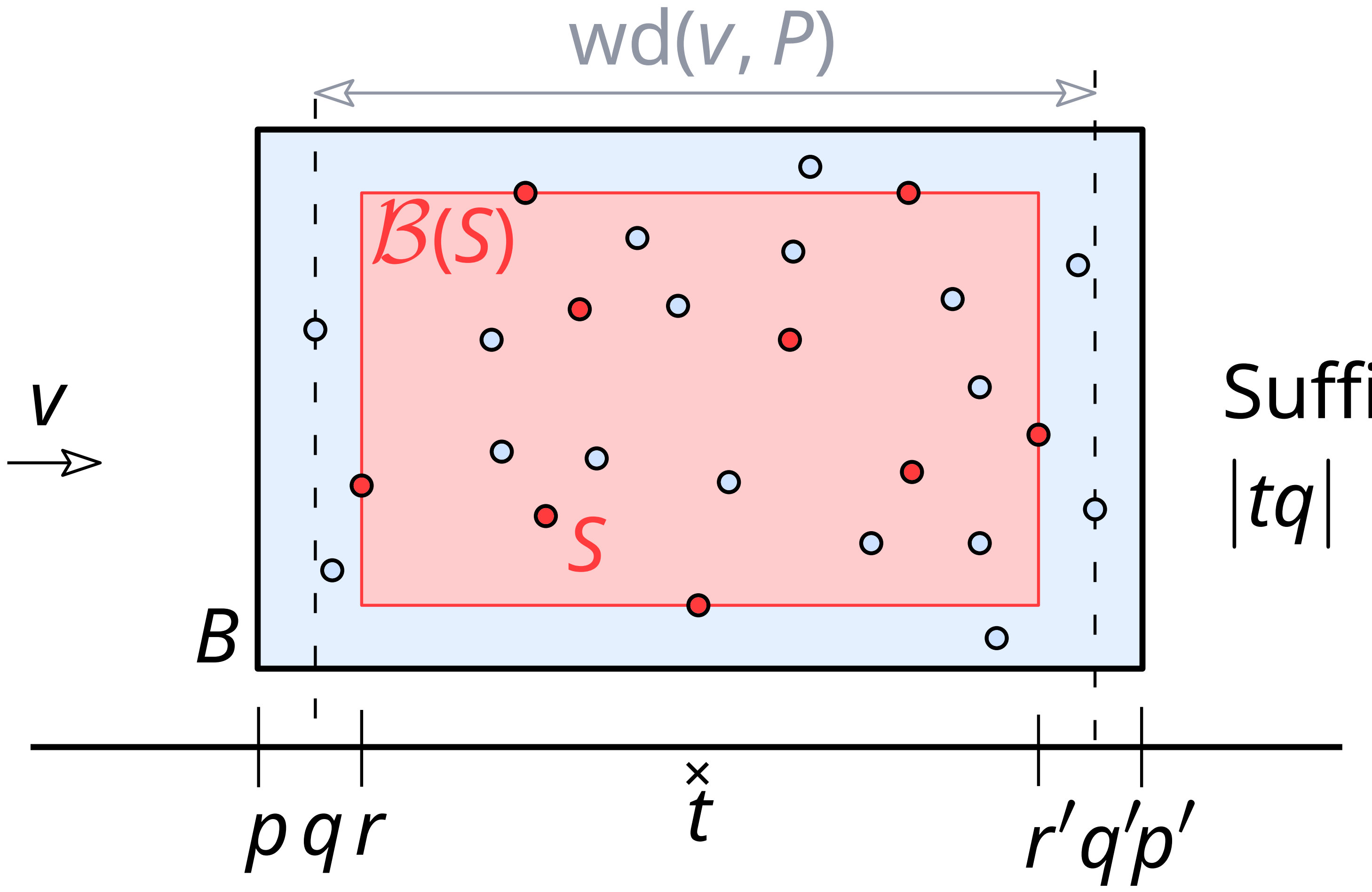

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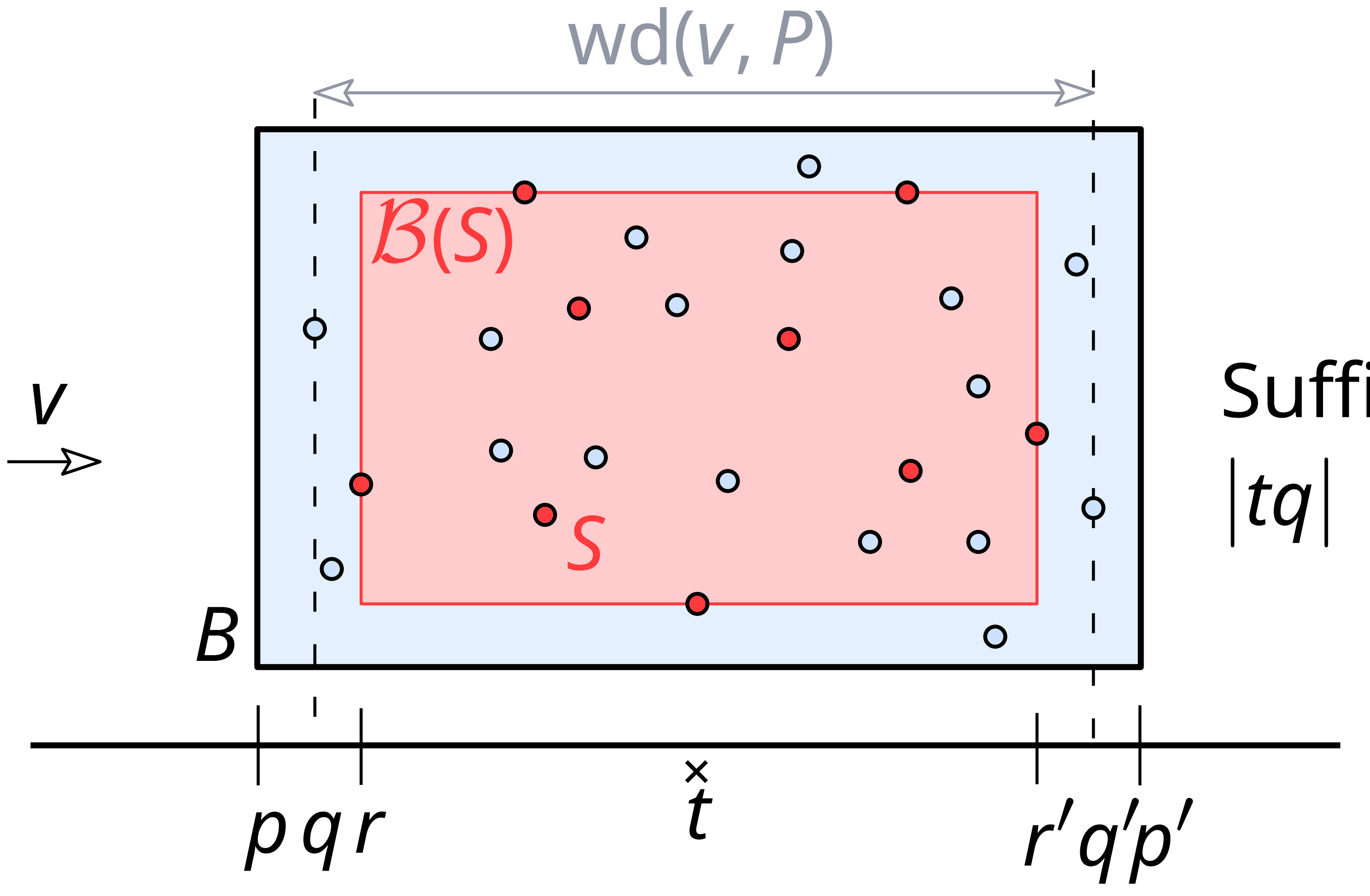

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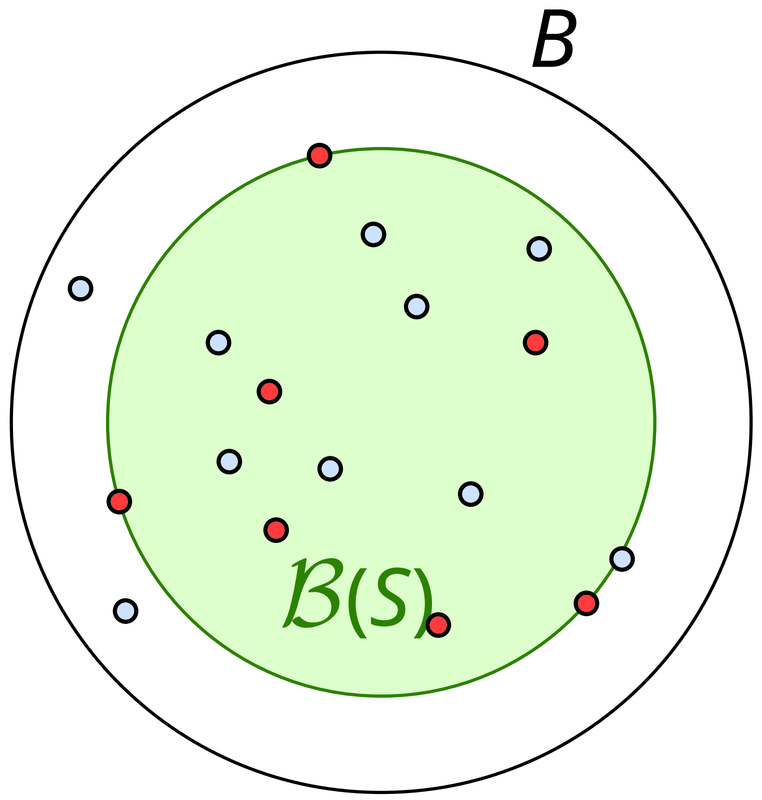
Use Case: minimum enclosing ball

If S is an $\varepsilon/4$-coreset of P for directional width, then the smallest enclosing ball of S, $\mathcal{B}(S)$, scaled by $(1+\varepsilon)$ around its center, B, contains P.



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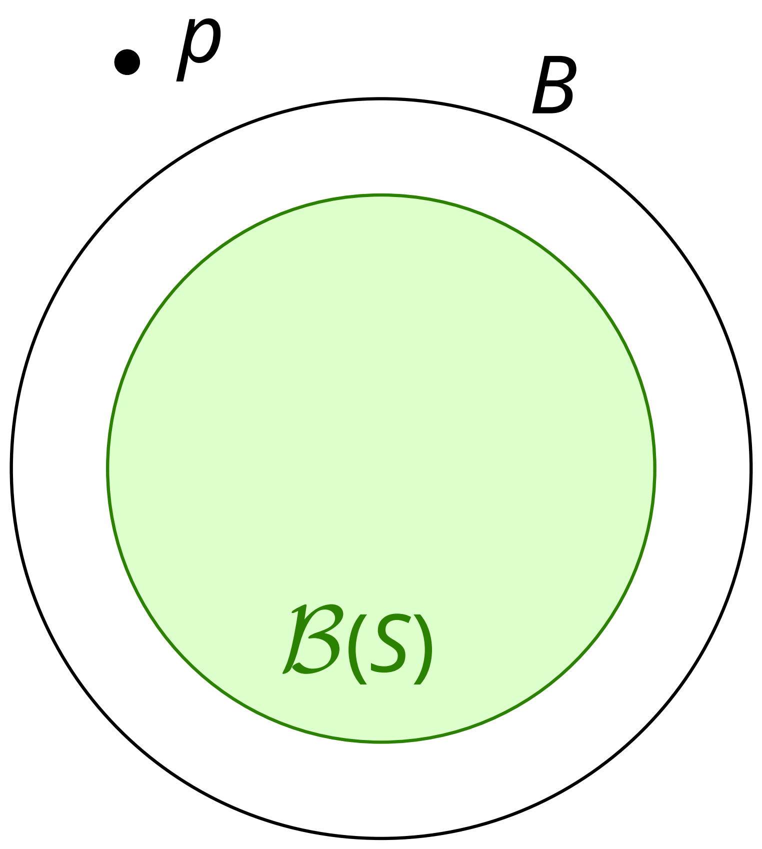

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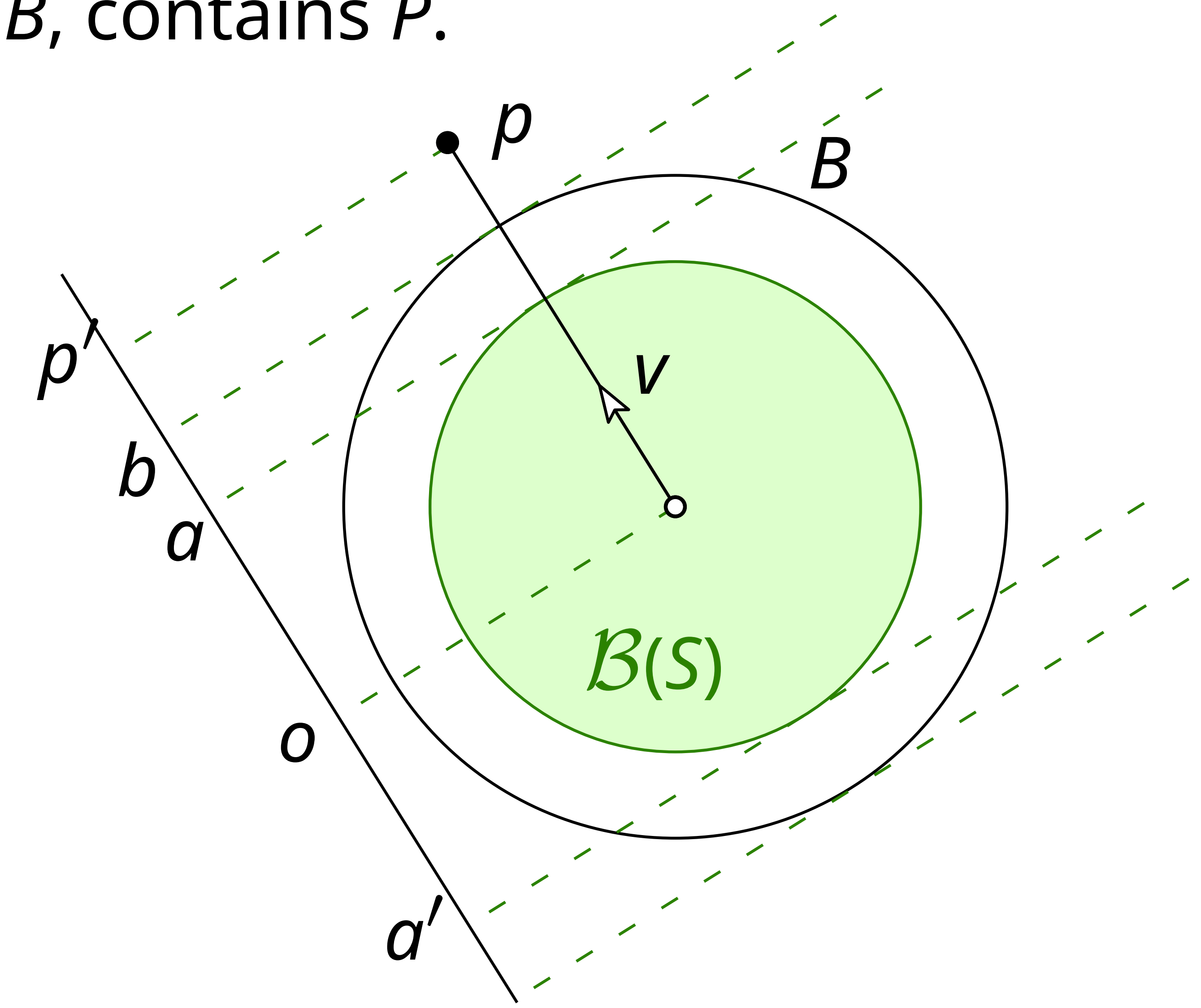

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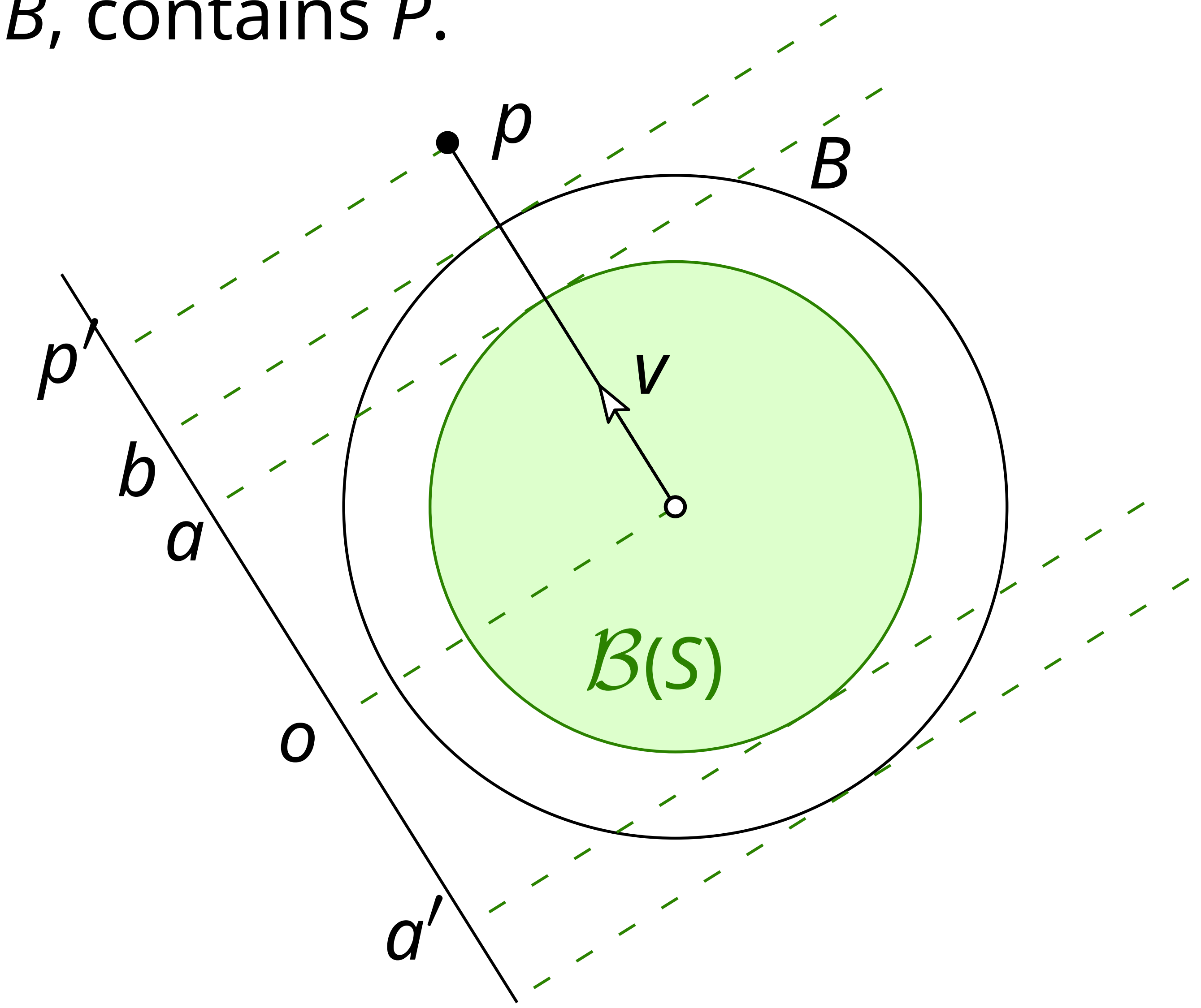

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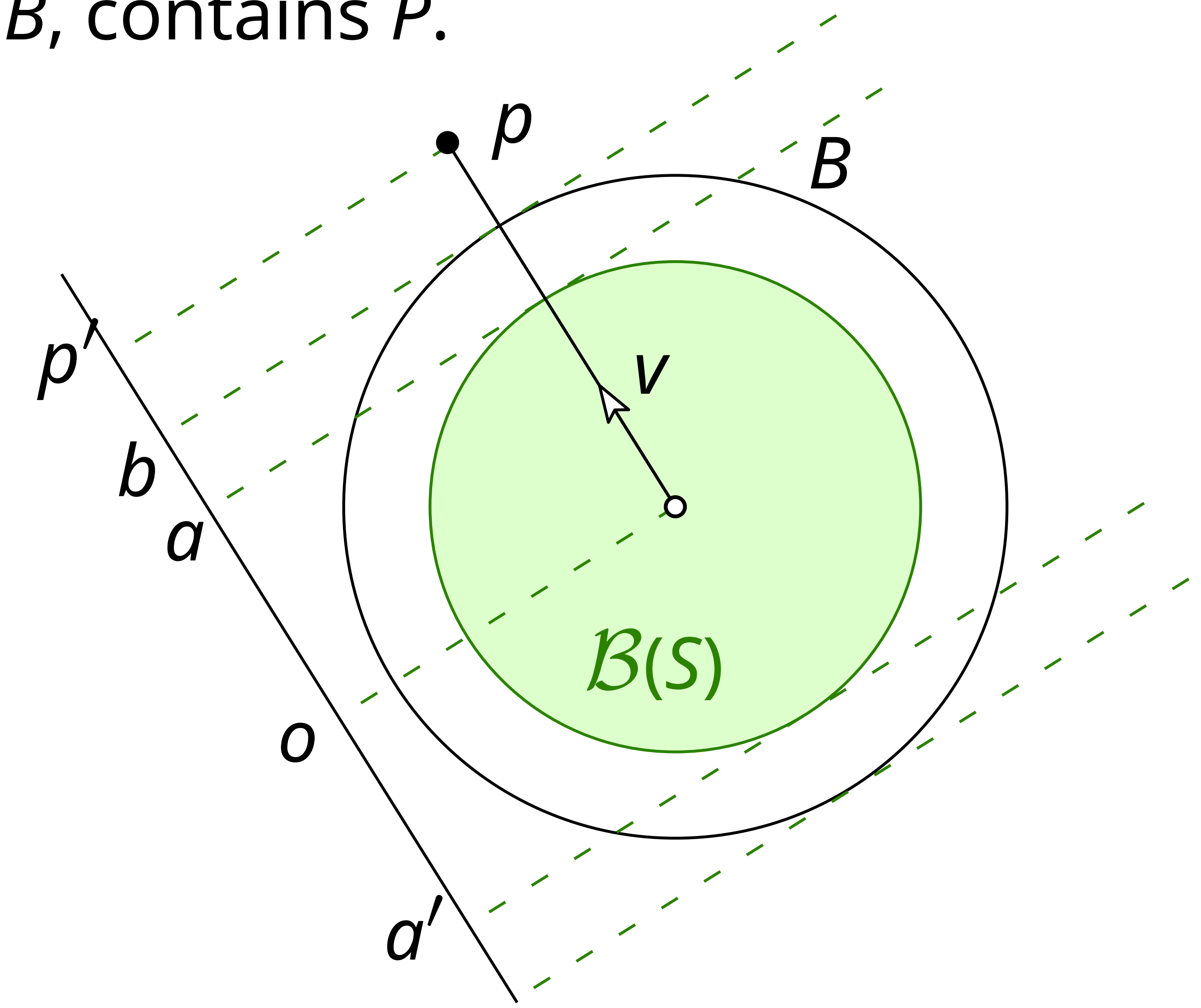

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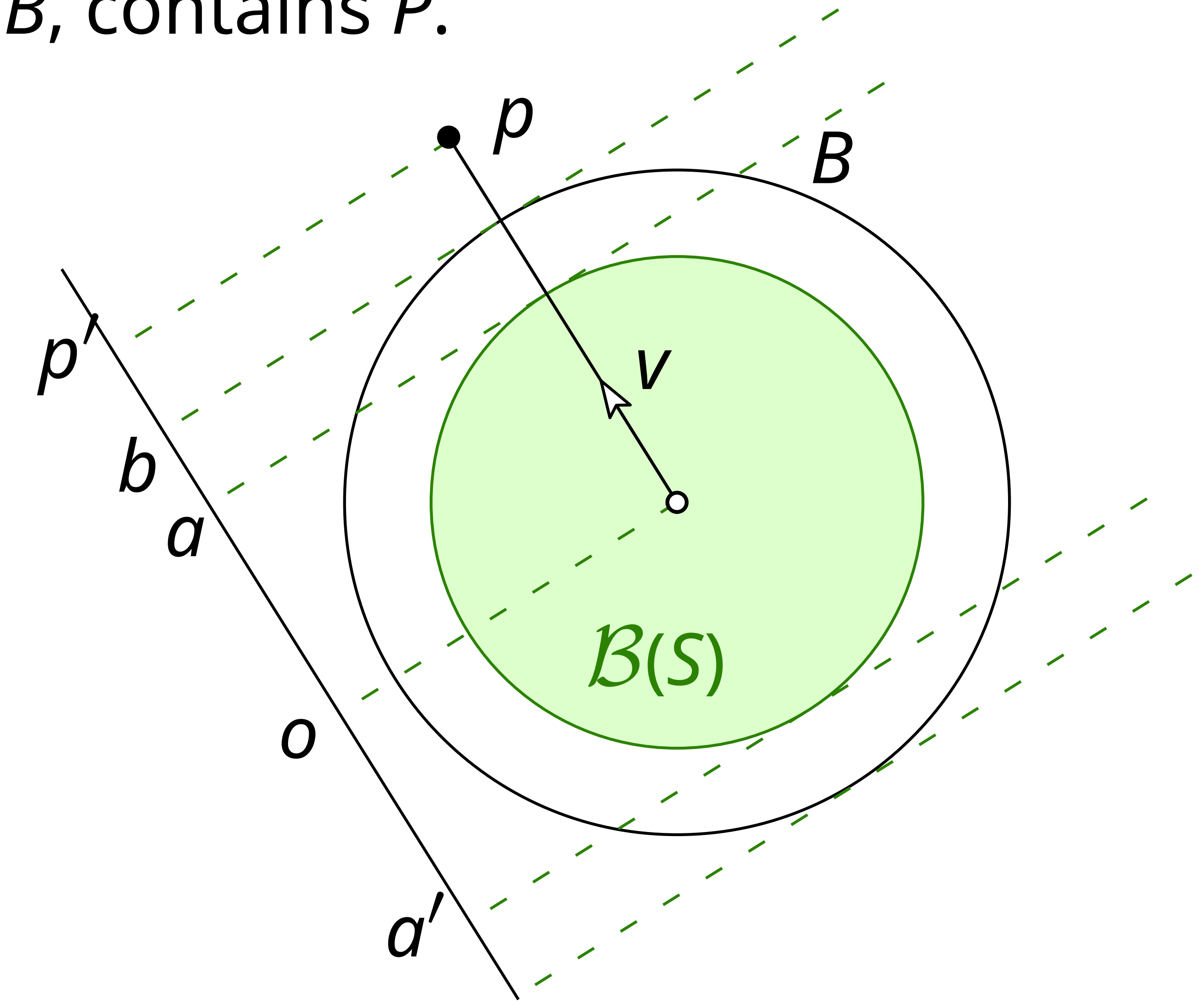

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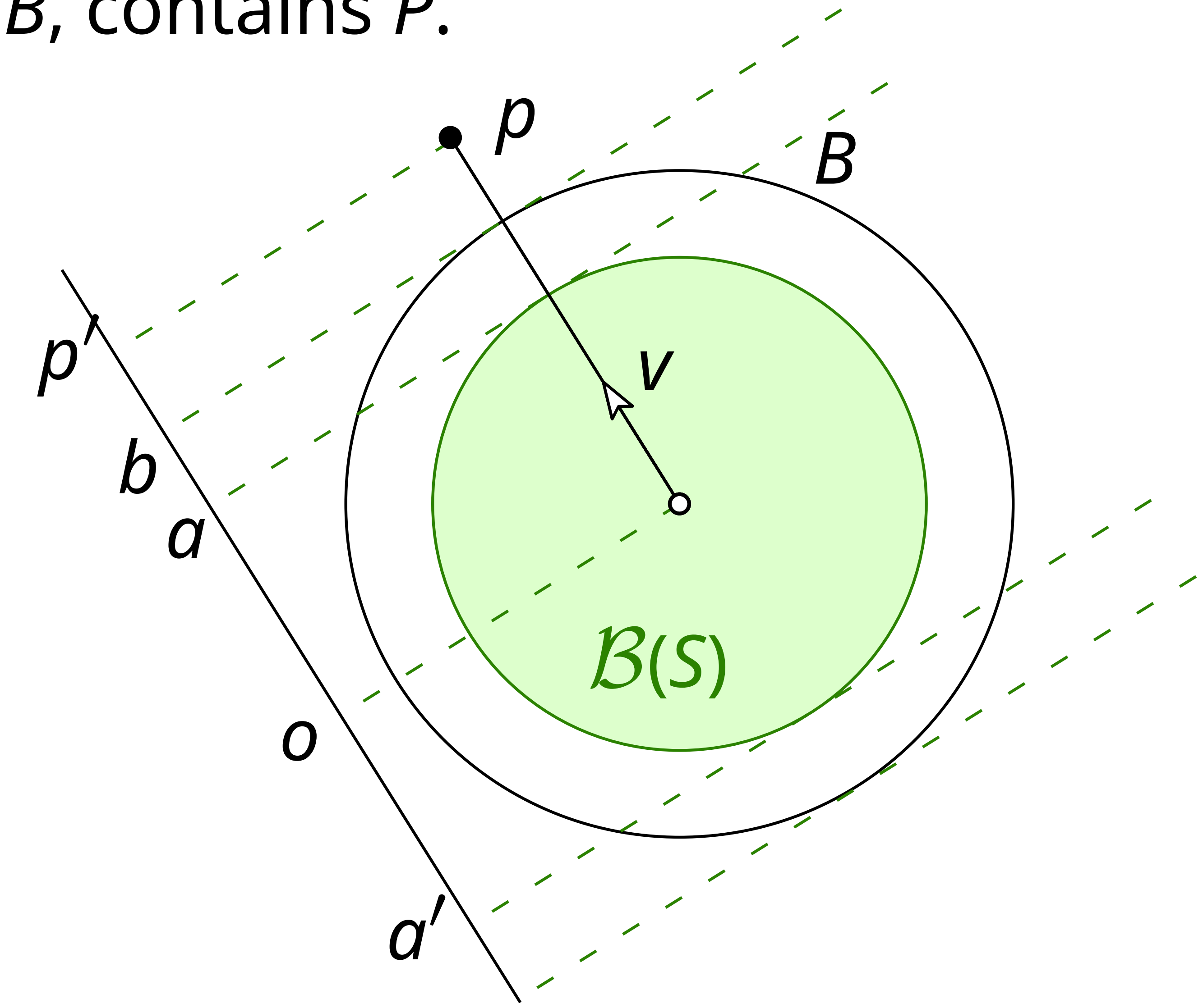

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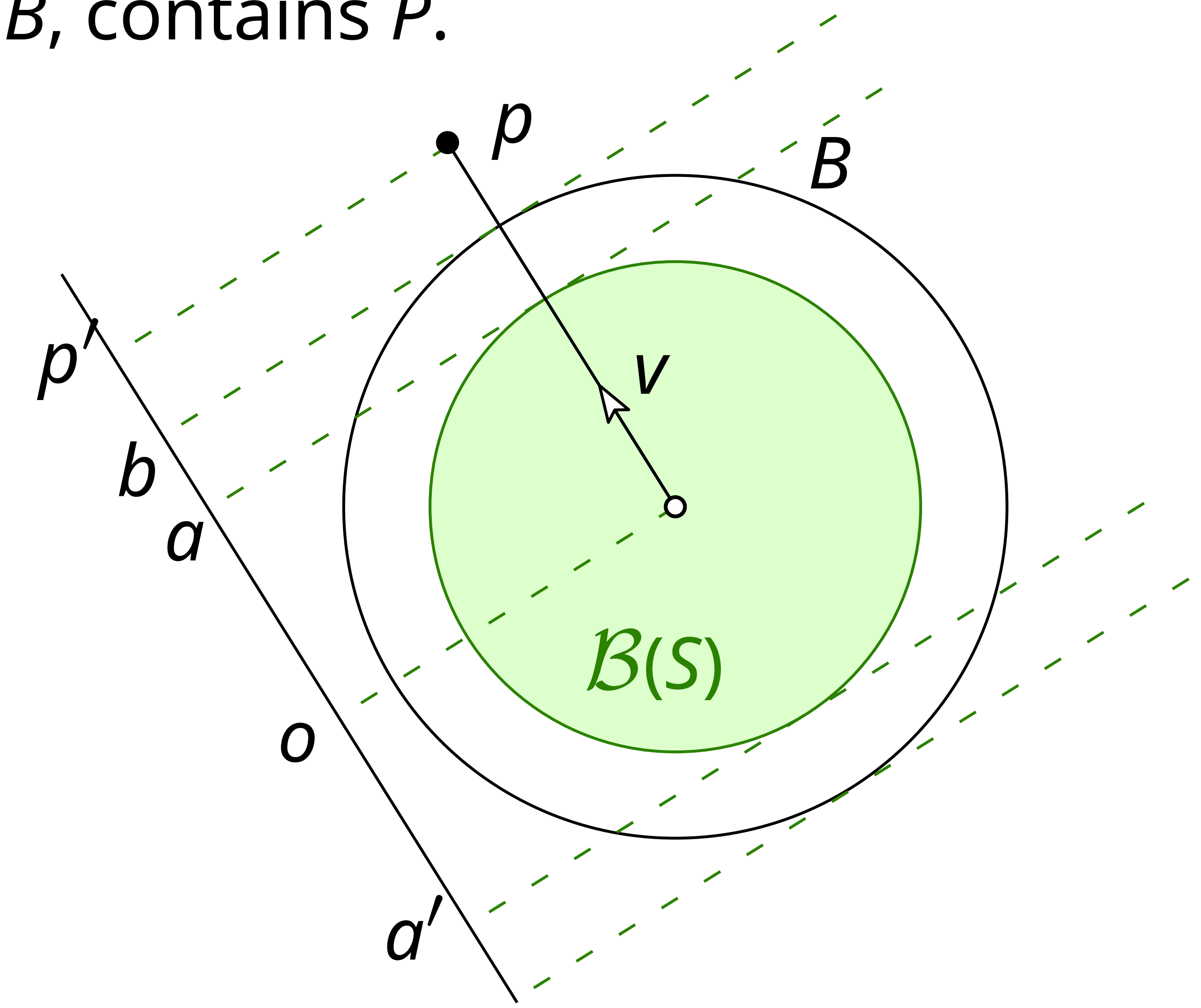

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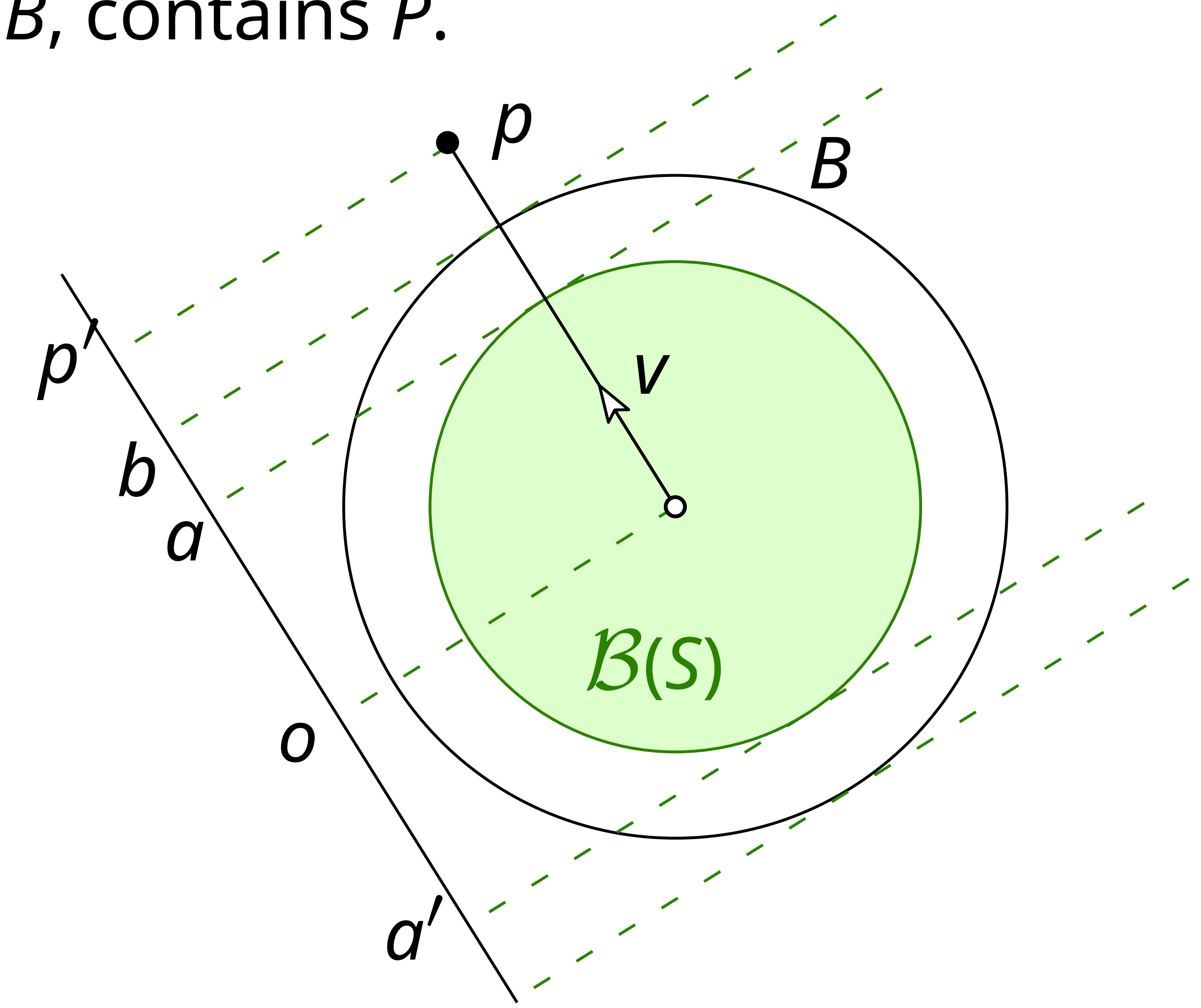

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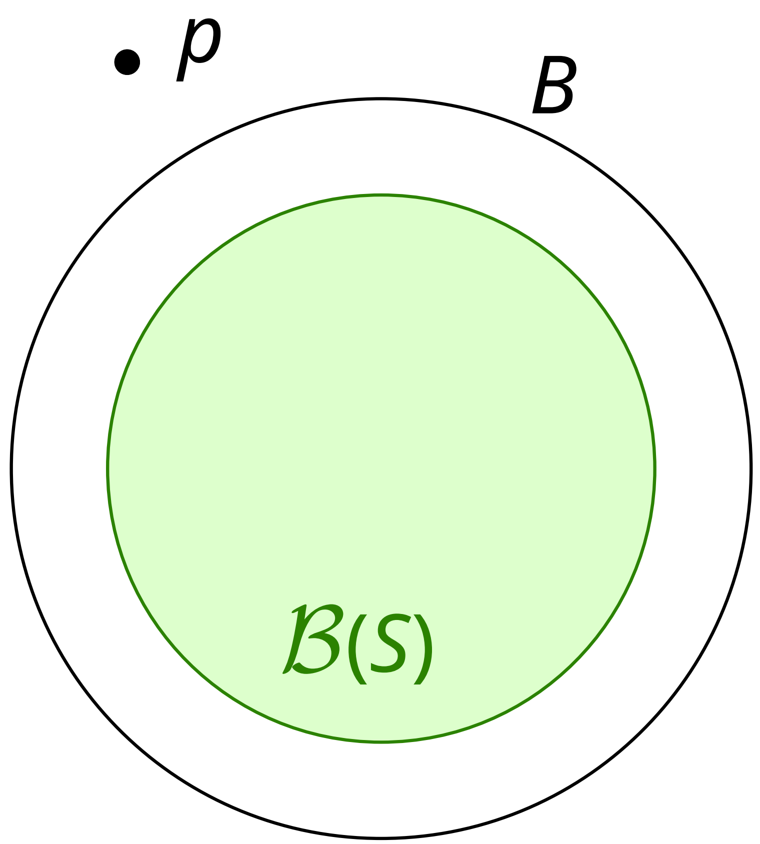

Comparison to ε-samples:

ε-sample:

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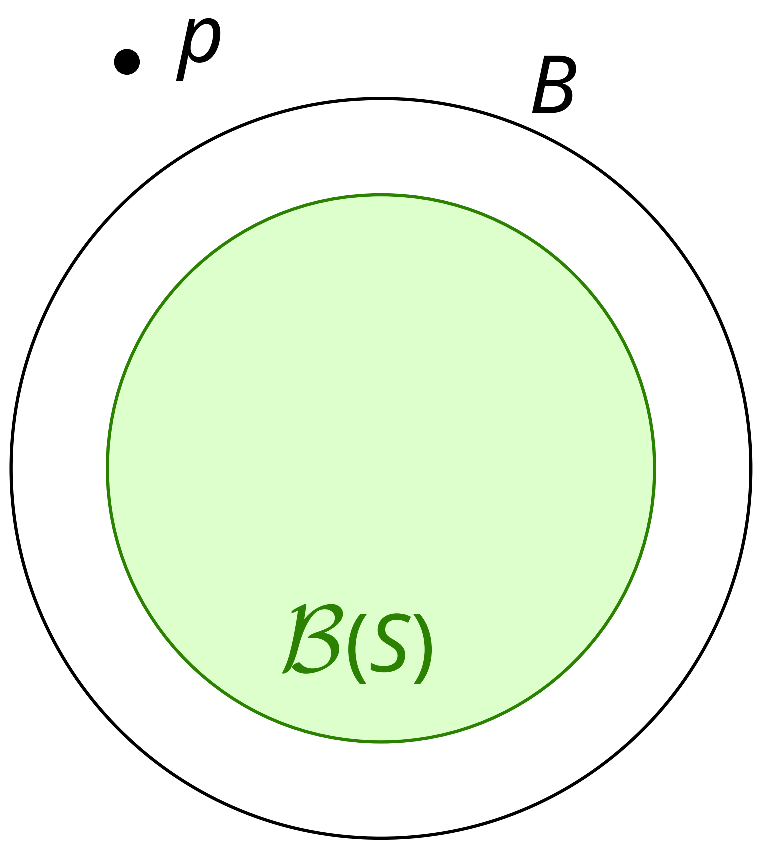

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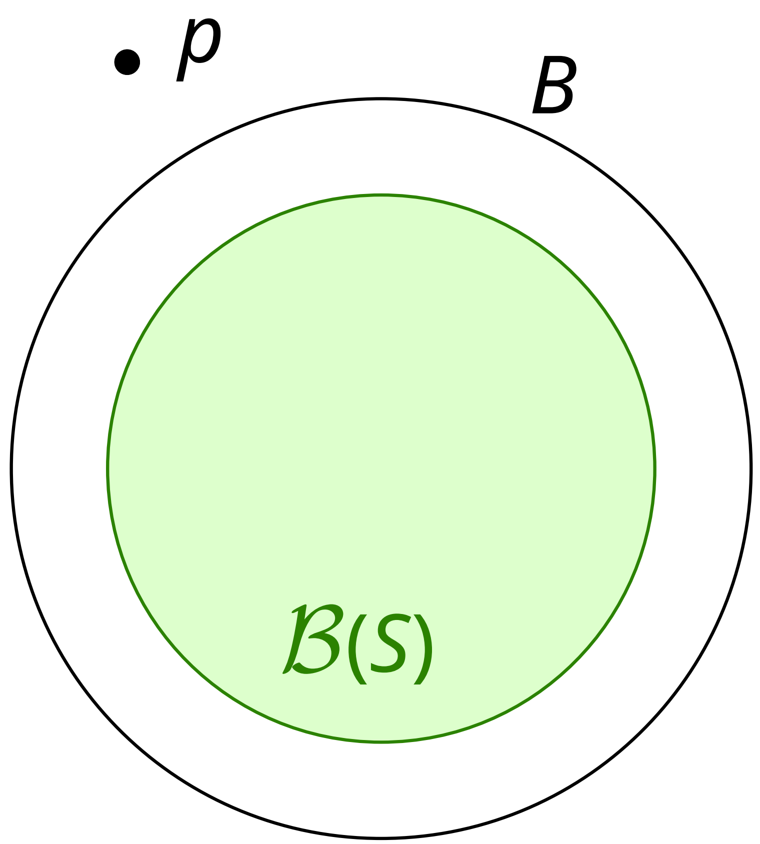

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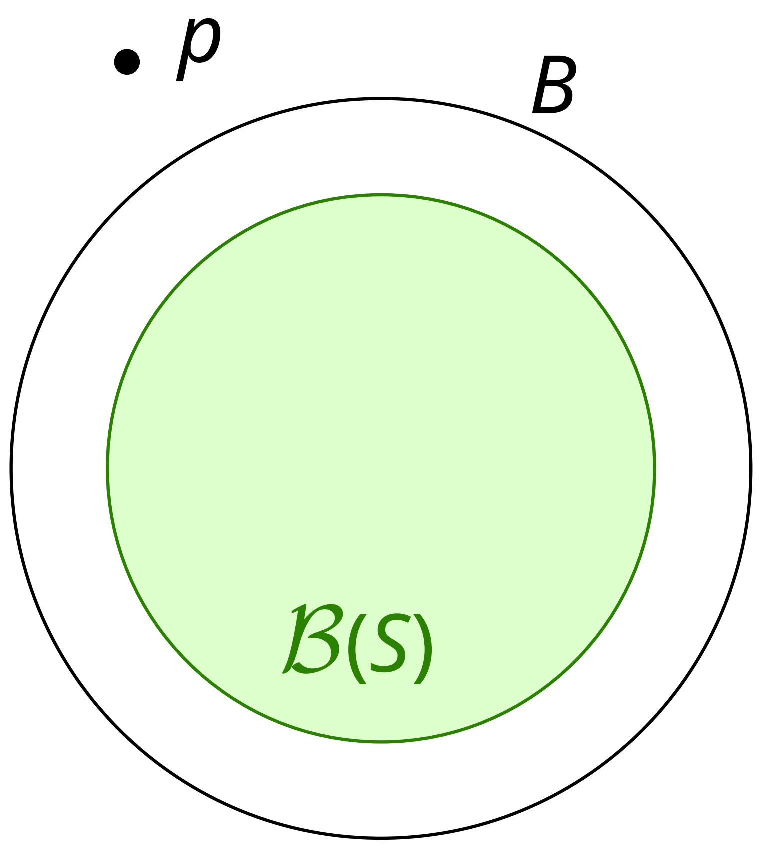

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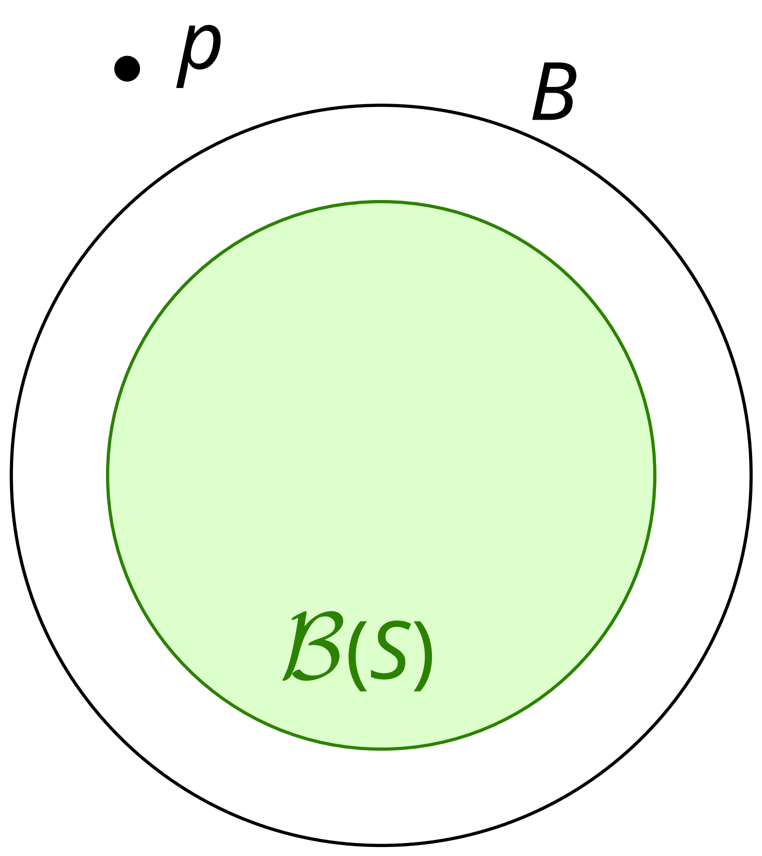

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coreset: geometric error (bounded for all points)

Overview

Coreset for directional width

- definition
- applications
- construction algorithm ←

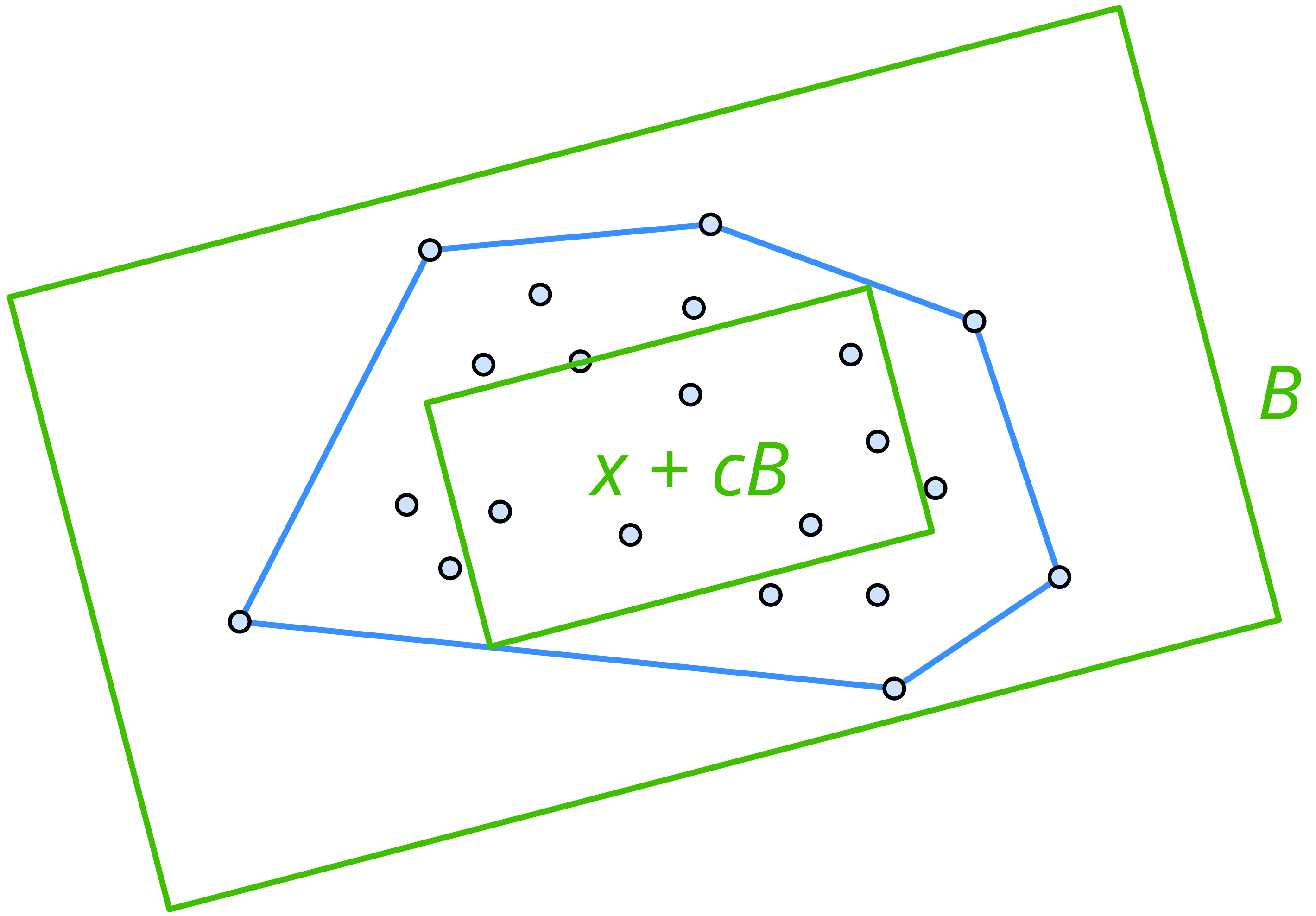
Extra ingredient: Minimum volume bounding box ←

Computing a tight (enough) bounding box

We can compute a bounding box B of P in $O(d^2 n)$ time s.t.

$$(i)\ \mathrm{Vol}(B_{opt}(P)) \leq \mathrm{Vol}(B) \leq 2^d d! \mathrm{Vol}(B_{opt}(P))$$

and (ii) there is a shift $x \in \mathbb{R}^d$ and $c > 0$ that depends only on d, s.t. $x + cB \subset \mathrm{conv}(P)$.



(without proof, for now)

Constructing a coreset

Input: $P \subset \mathbb{R}^d$, $\varepsilon > 0$ (and bounding box B s.t. $c_d B \subset \text{conv}(P) \subset B$)

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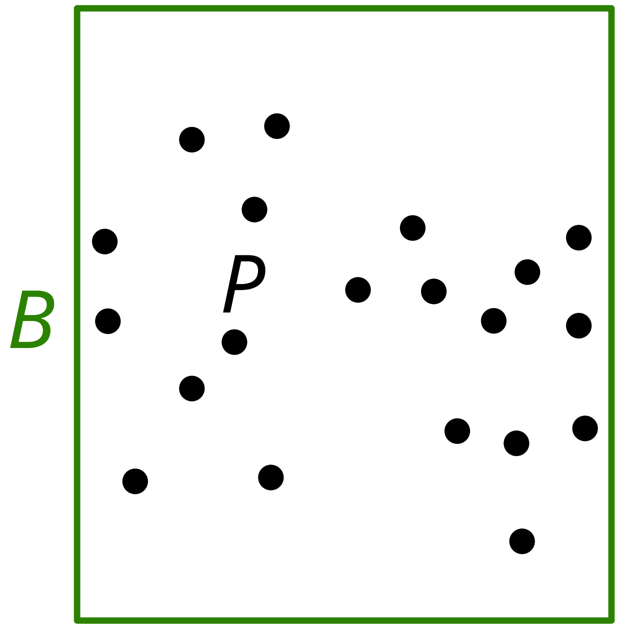
Construction time: $O(n)$ (also depends on d and ε).



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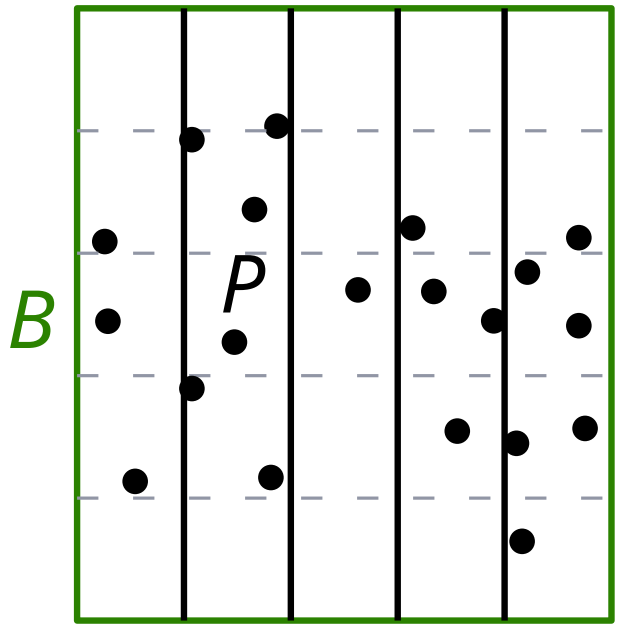

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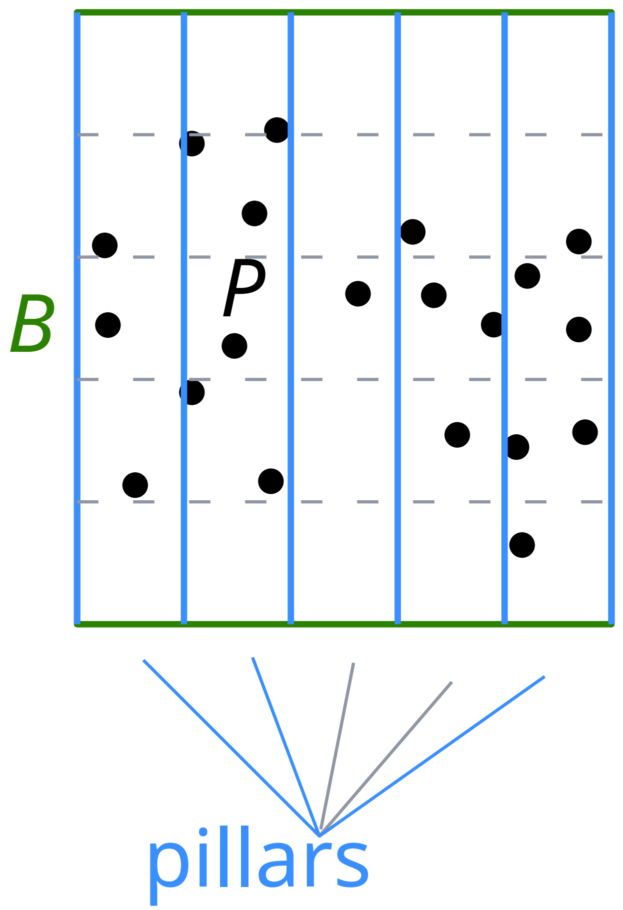

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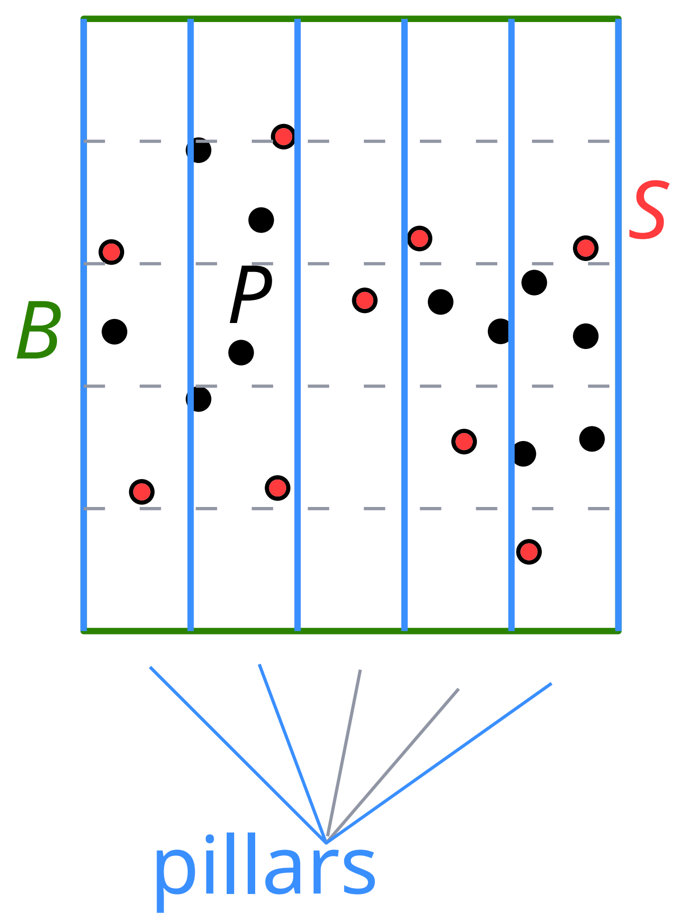

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$\underbrace{\hphantom{xxxxxxxxxx}}_{S}$

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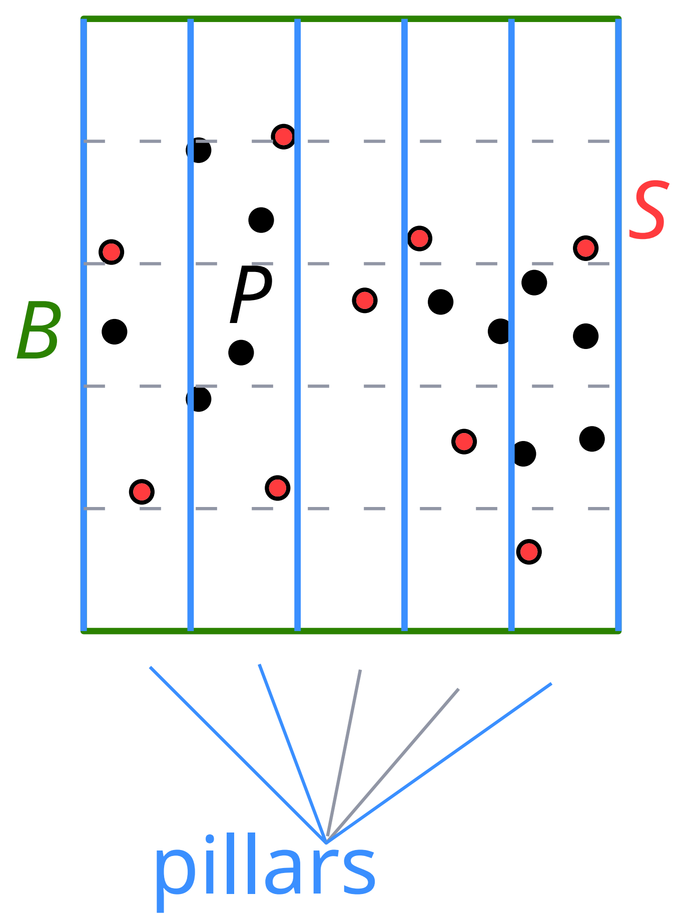

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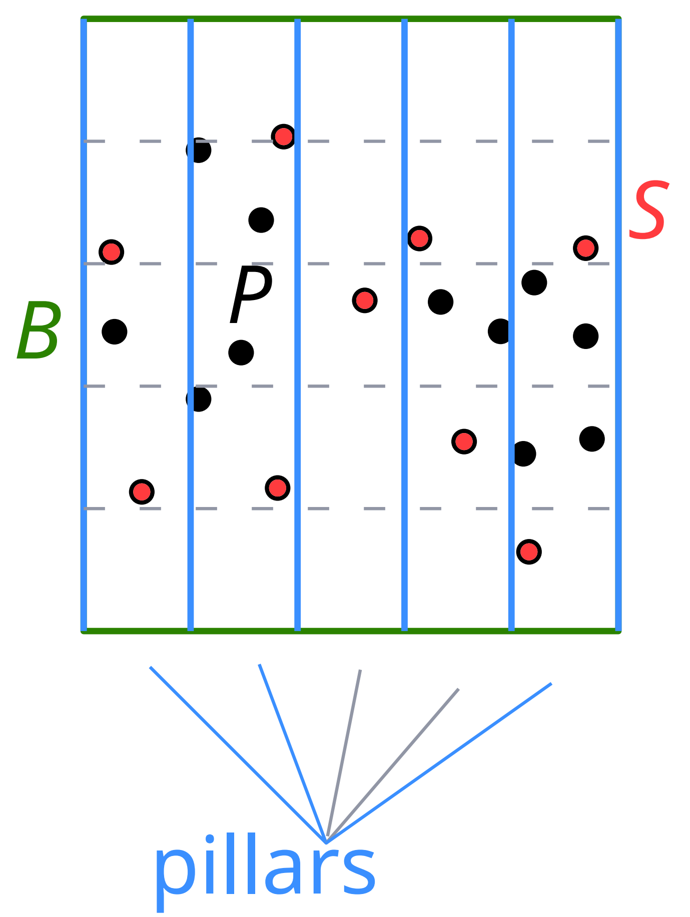

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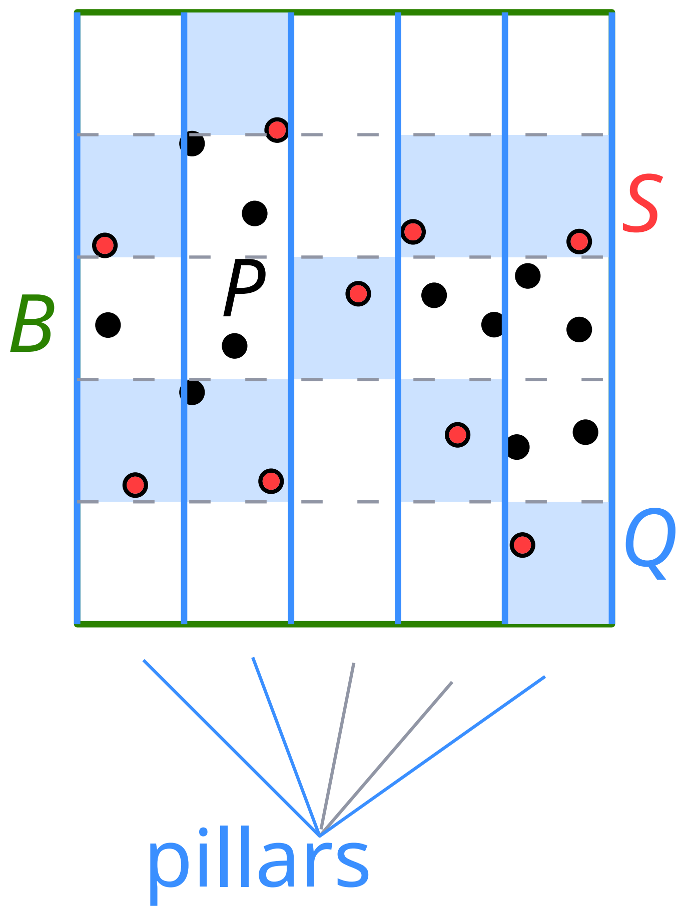

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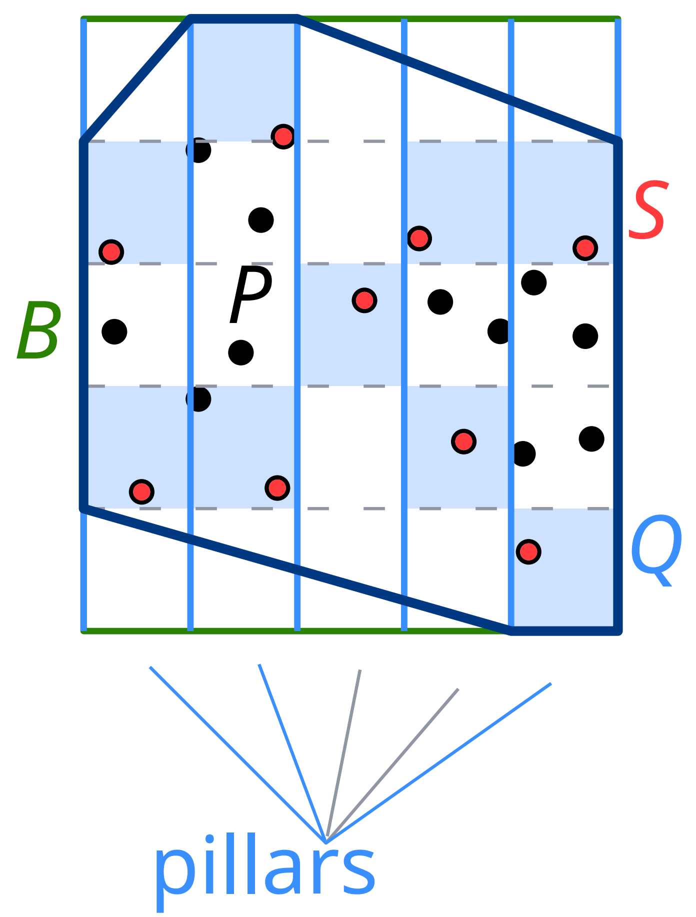

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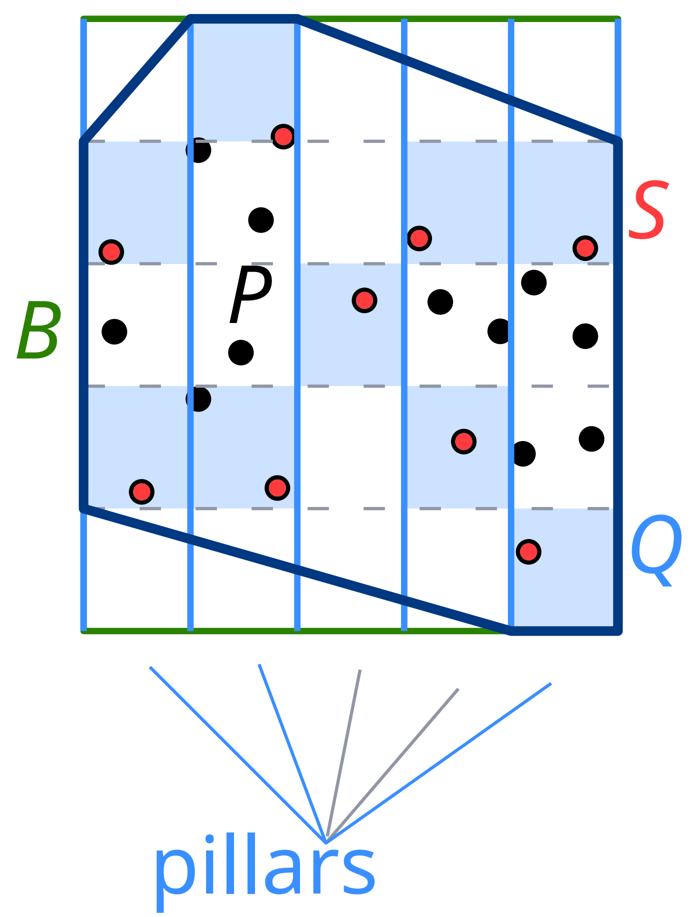

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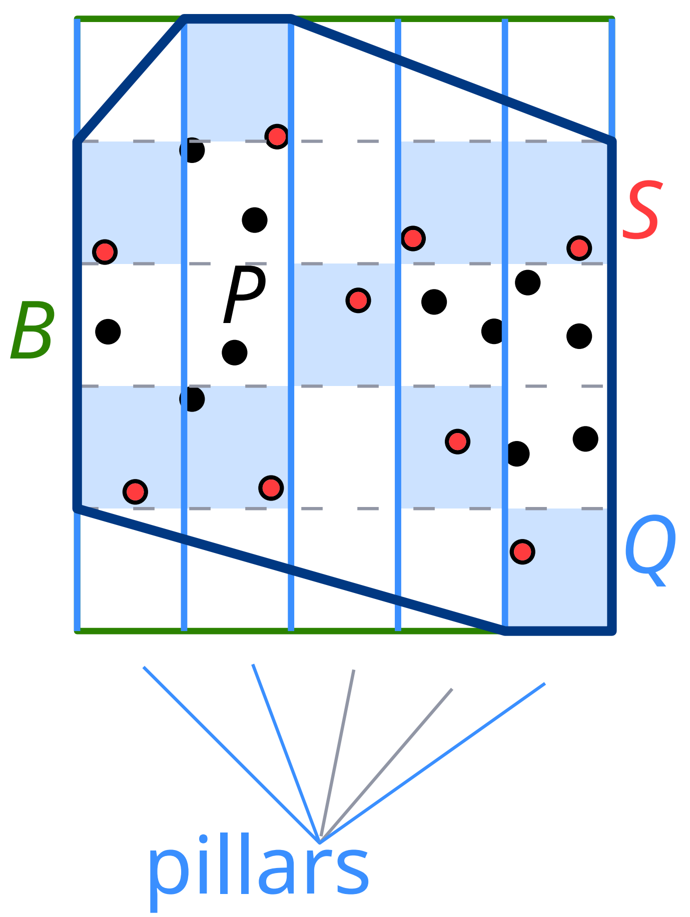

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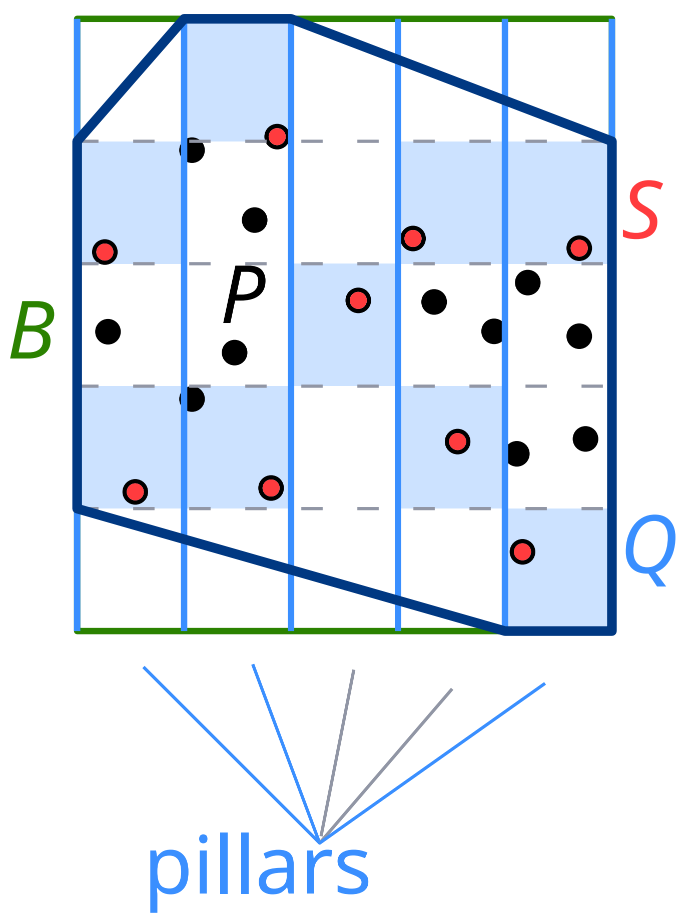

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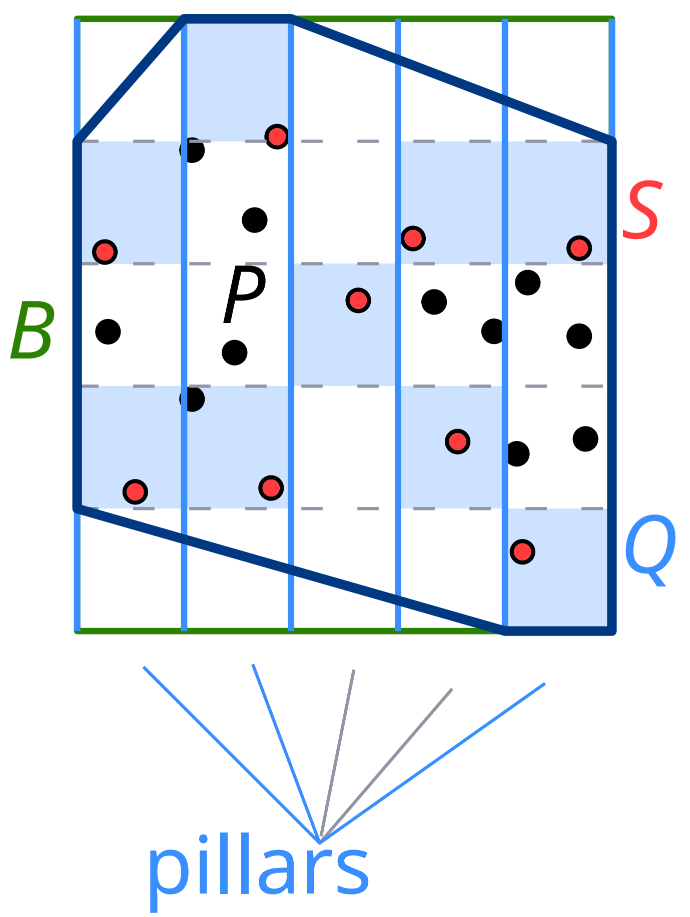
□



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Given $\varepsilon > 0$ and $P \subset \mathbb{R}^d$, we can compute an ε-coreset $S \subseteq P$ of size at most $|S| = O(1/\varepsilon^{(d-1)/2})$ in $O(n + 1/\varepsilon^{3(d-1)/2})$ time (where d is a fixed constant).

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algorithm ideas/proof sketch:

- two stages: first the previous algo. for $\varepsilon/2$ gives S', then this (slower) algorithm for $\varepsilon/2$ on S' gives S
- make conv(P) fat via affine transformation into unit hypercube
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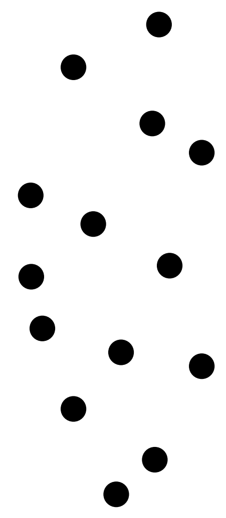
- Let S = nearest point to each $x \in X$



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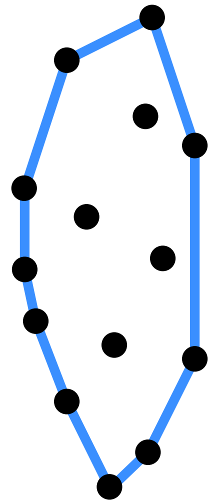

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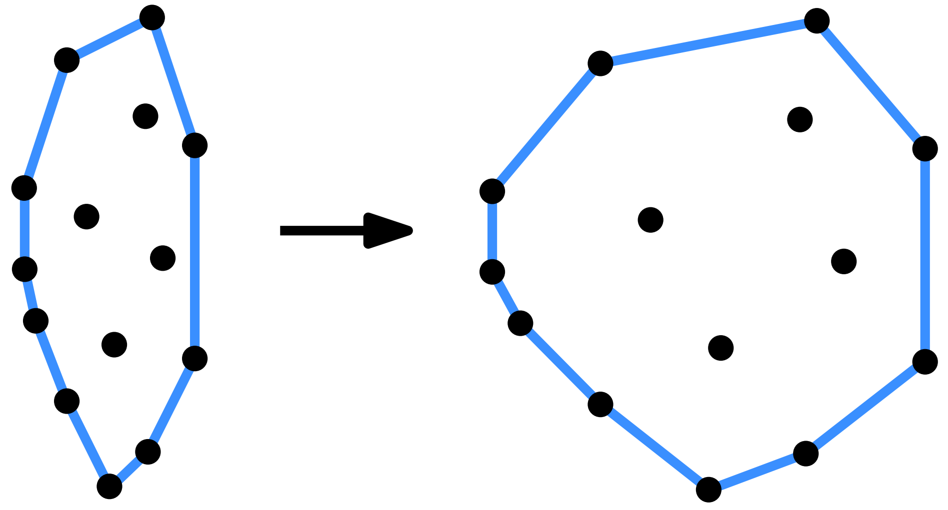

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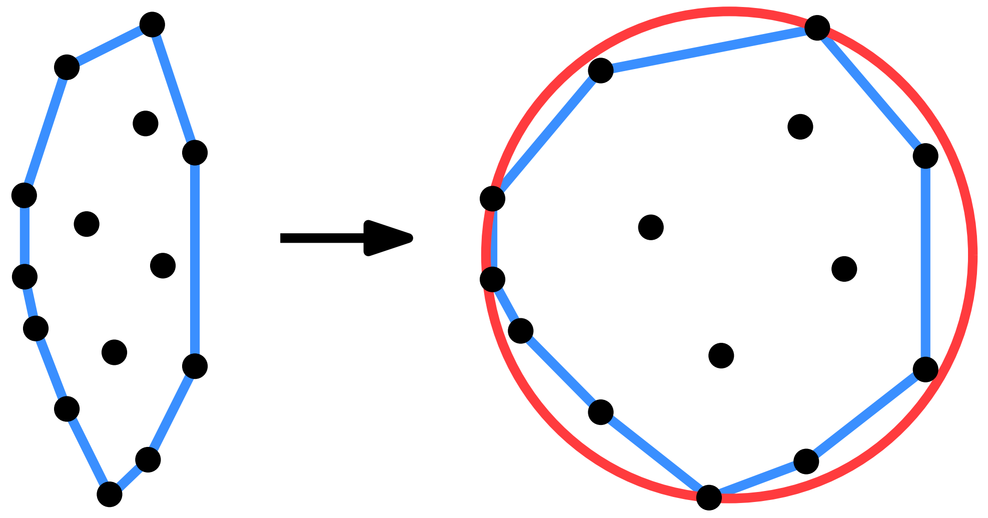

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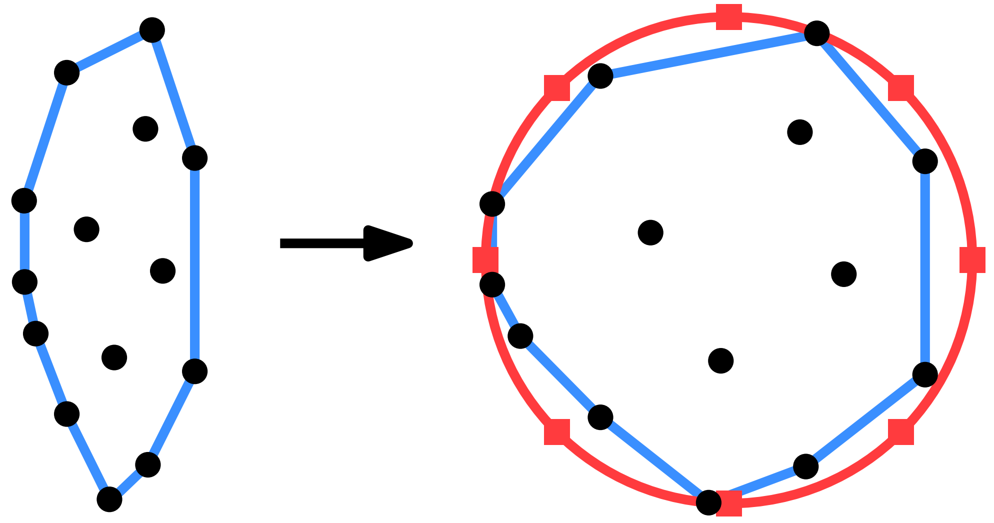

Constructing a smaller coreset

Given $\varepsilon > 0$ and $P \subset \mathbb{R}^d$, we can compute an ε-coreset $S \subseteq P$ of size at most $|S| = O(1/\varepsilon^{(d-1)/2})$ in $O(n + 1/\varepsilon^{3(d-1)/2})$ time (where d is a fixed constant).

algorithm ideas/proof sketch:

- two stages: first the previous algo. for $\varepsilon/2$ gives S', then this (slower) algorithm for $\varepsilon/2$ on S' gives S
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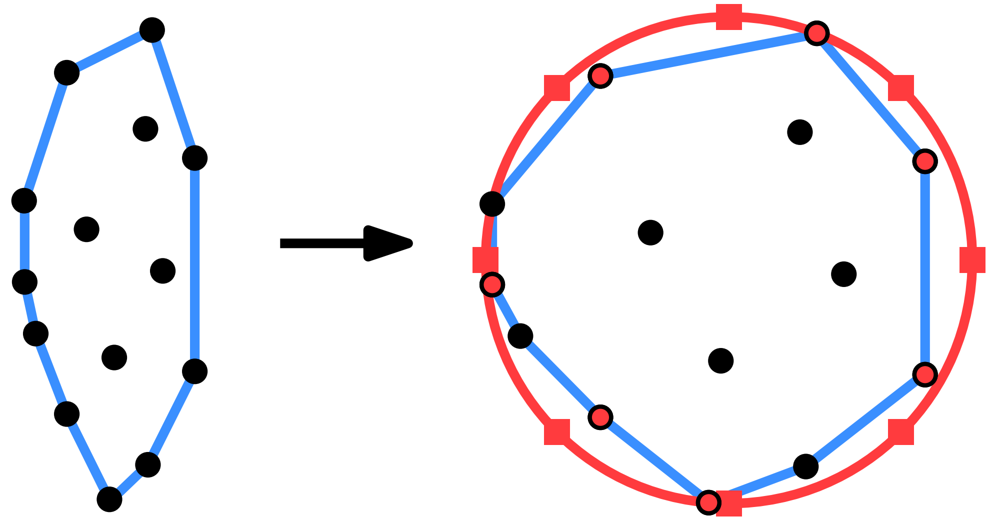

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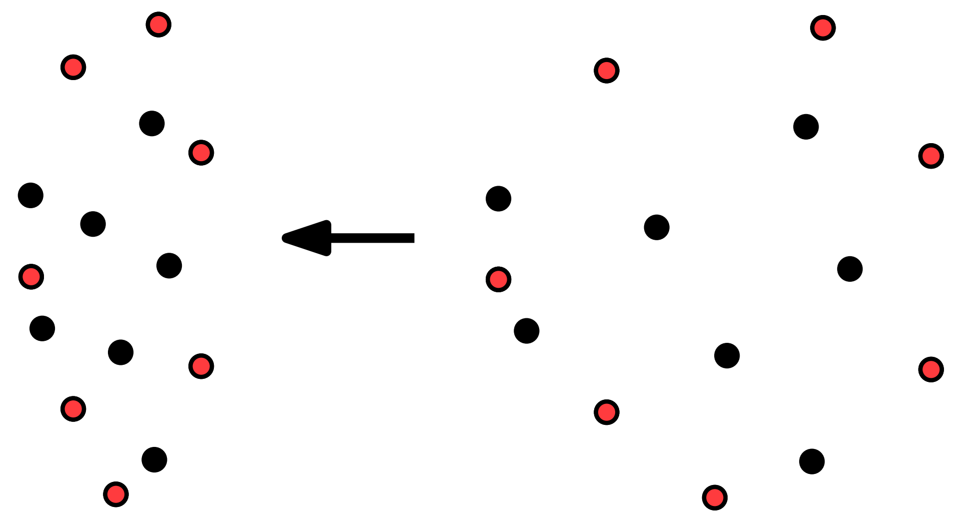


Overview

Coreset for directional width

- definition
- applications
- construction algorithm

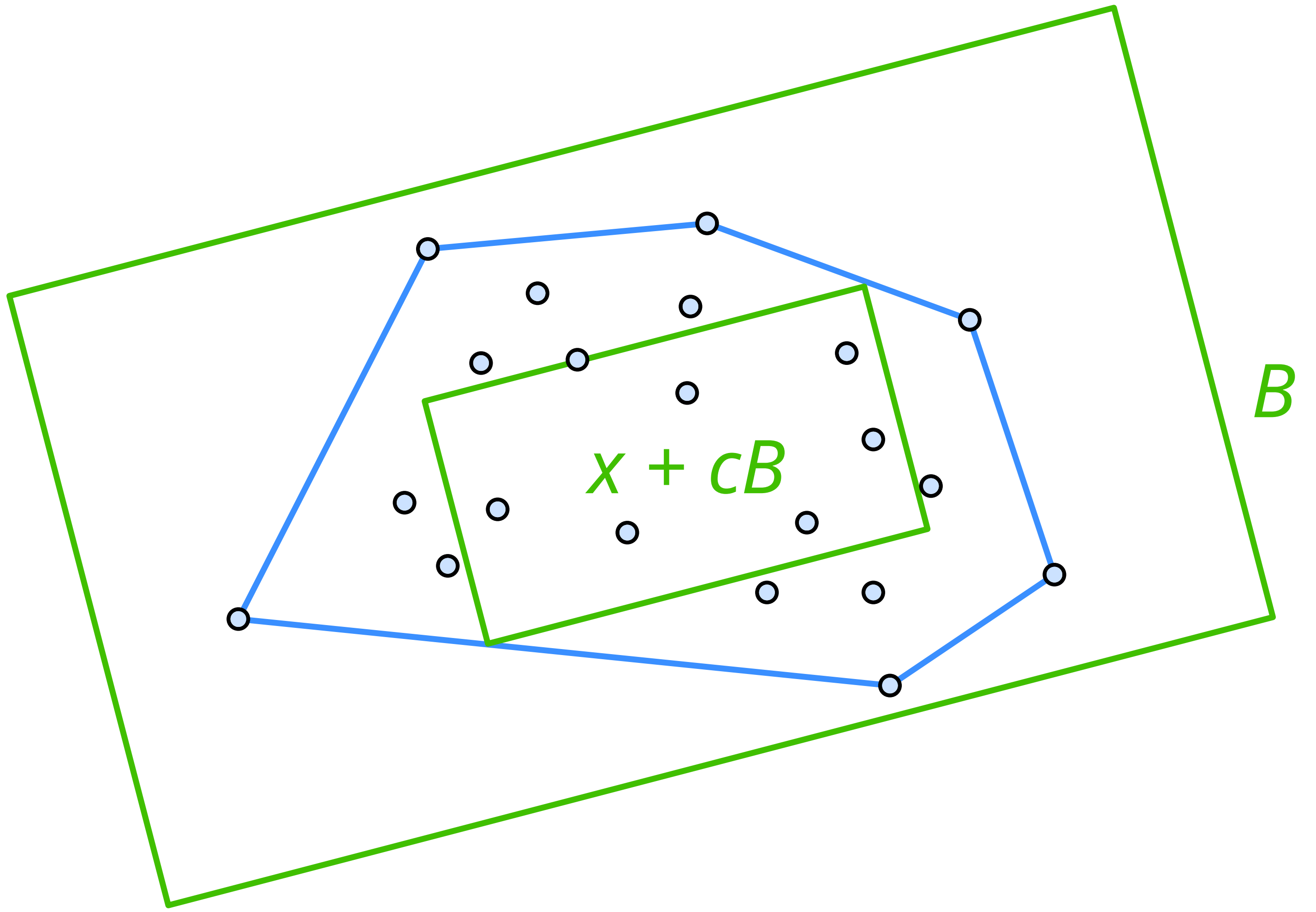
Extra ingredient: Minimum volume bounding box ←

Computing a tight (enough) bounding box

We can compute a bounding box B of P in $O(d^2 n)$ time s.t.

$$(i)\ \text{Vol}(B_{opt}(P)) \leq \text{Vol}(B) \leq 2^d d!\text{Vol}(B_{opt}(P))$$

and (ii) there is a shift $x \in \mathbb{R}^d$ and $c > 0$ that depends only on d, s.t. $x + cB \subset \text{conv}(P)$.



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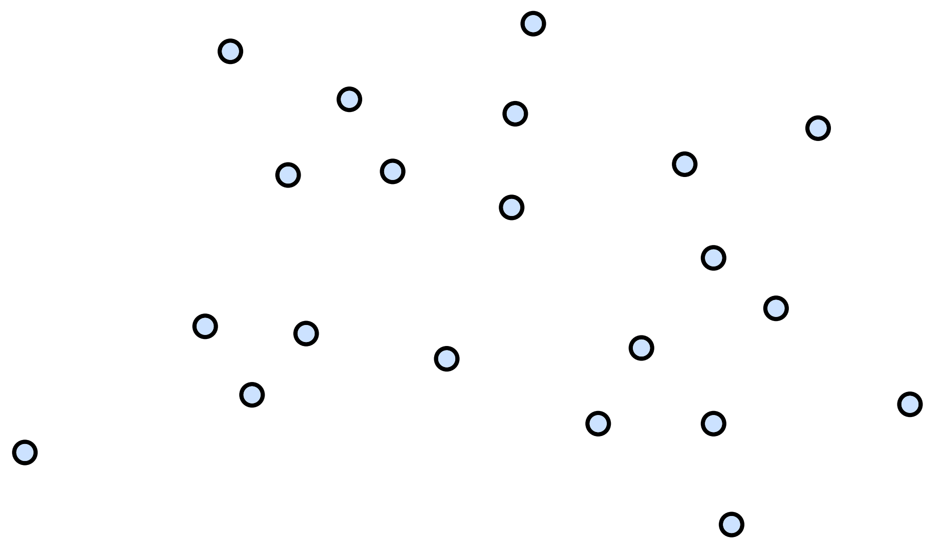

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If t, t' realize the diameter of P, then

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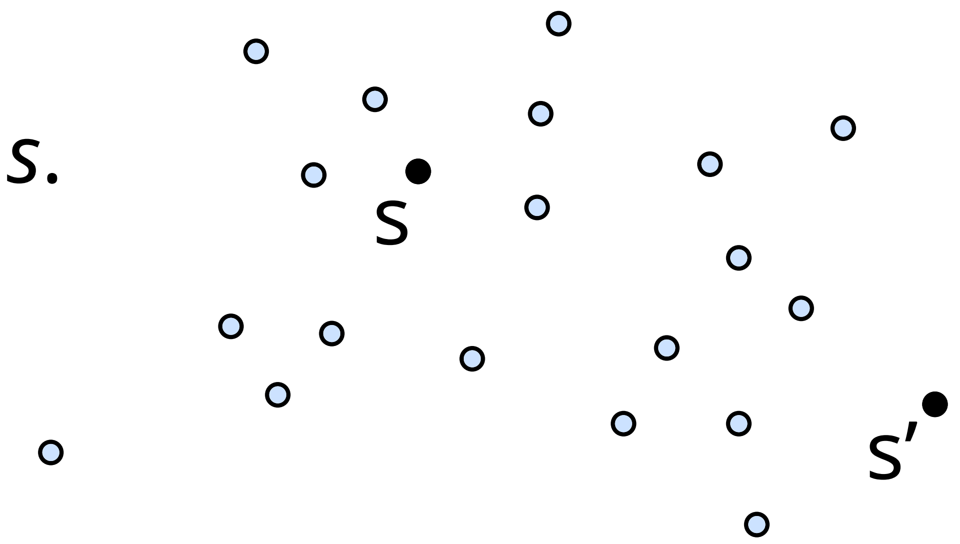

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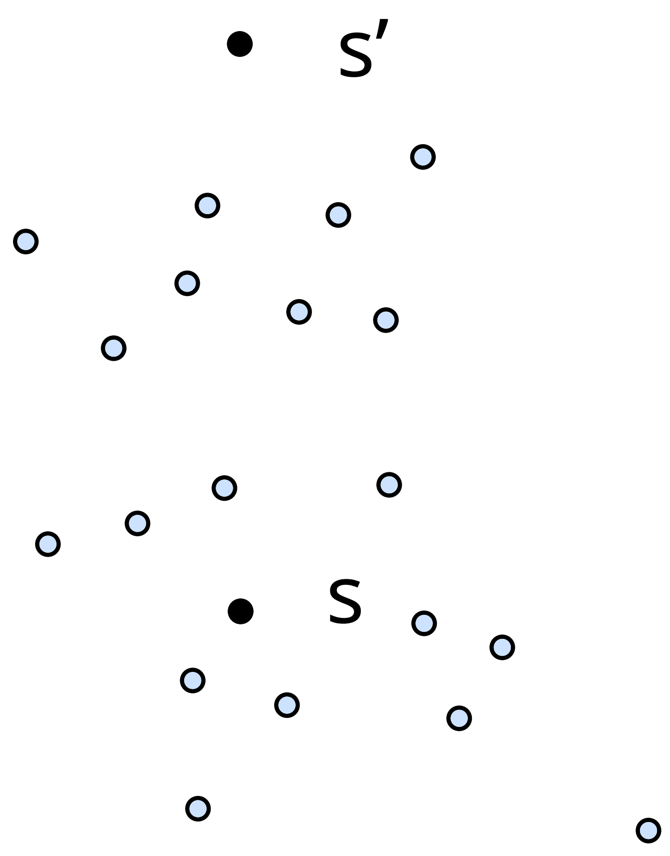
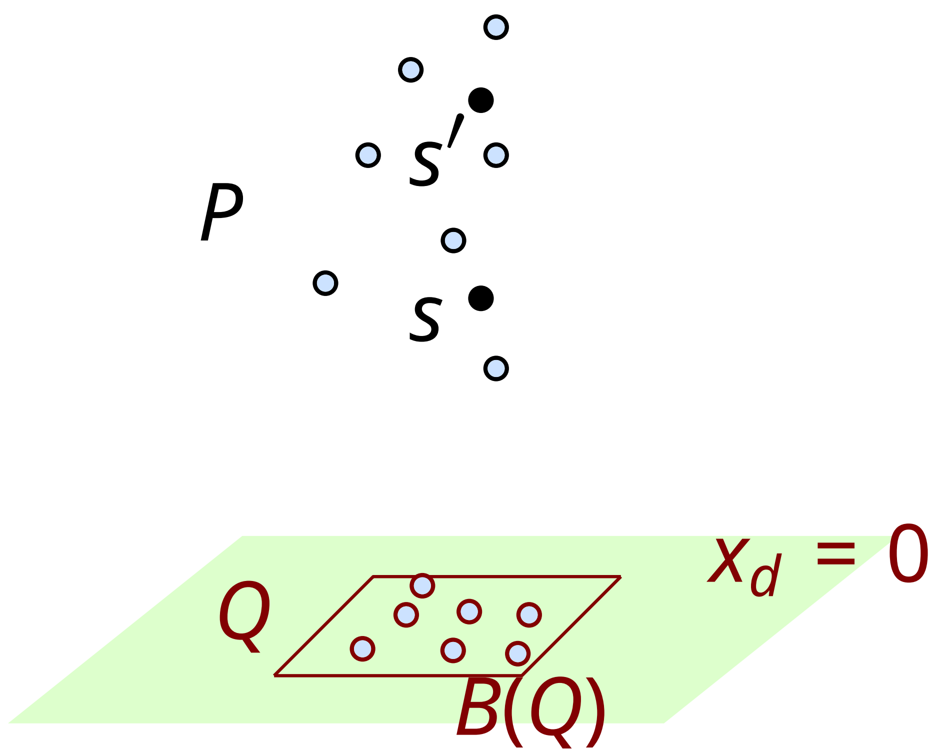
Wlog. ss' parallel to x_d axis.
π := perpendicular projection to $x_d = 0$.

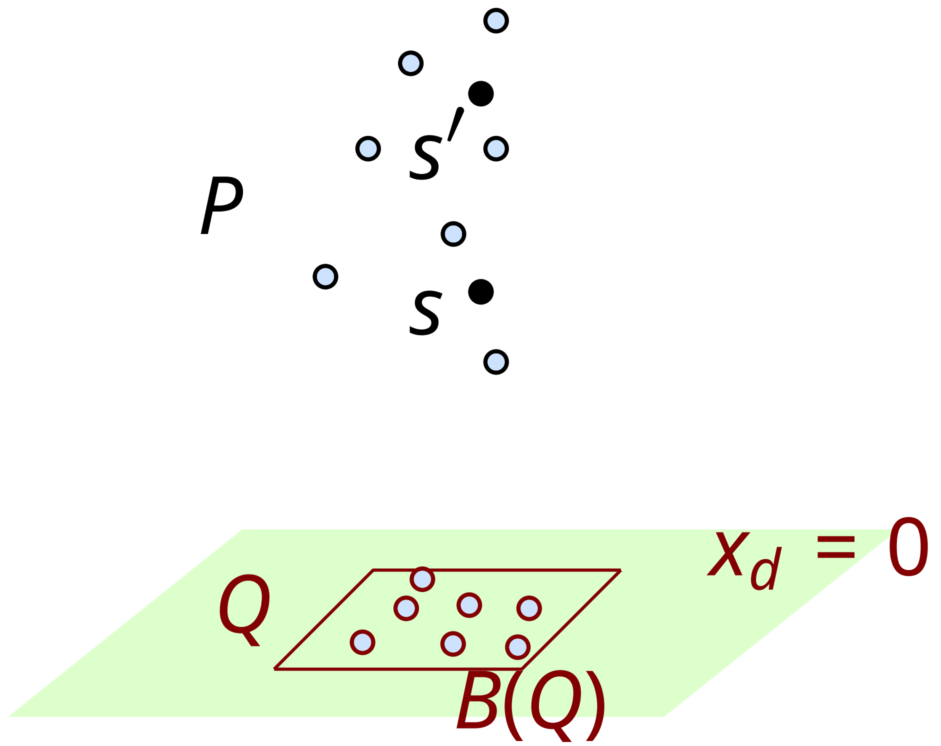


Recursive step



Recursive step

$d = 1$: return interval containing points



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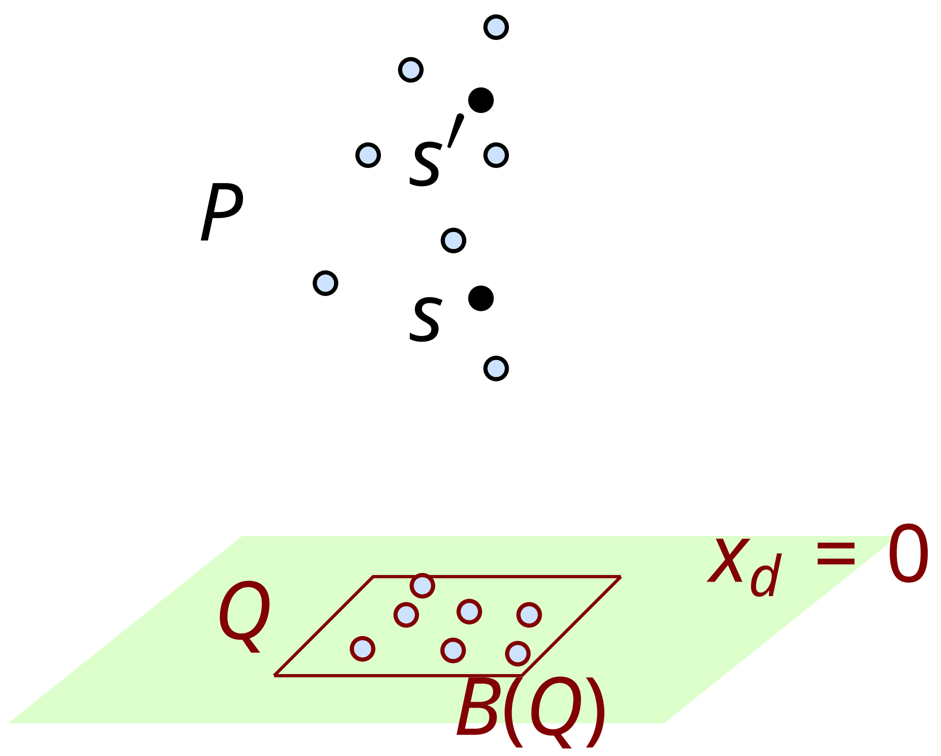

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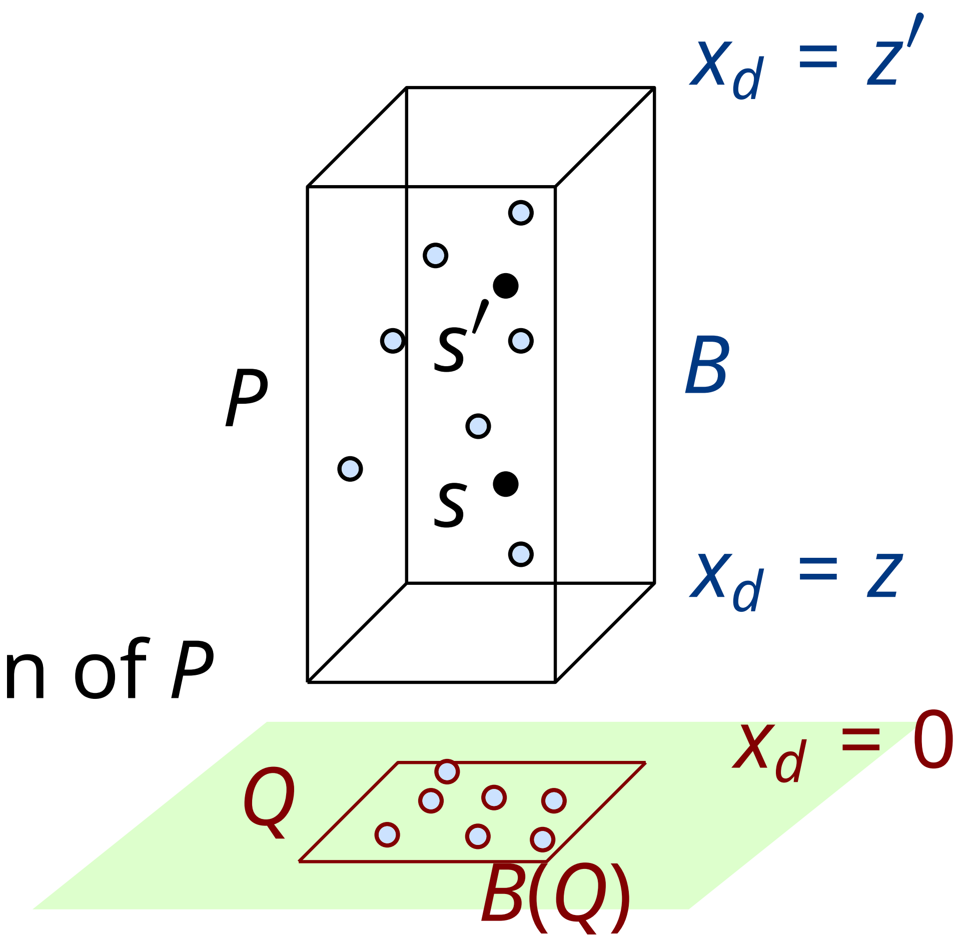

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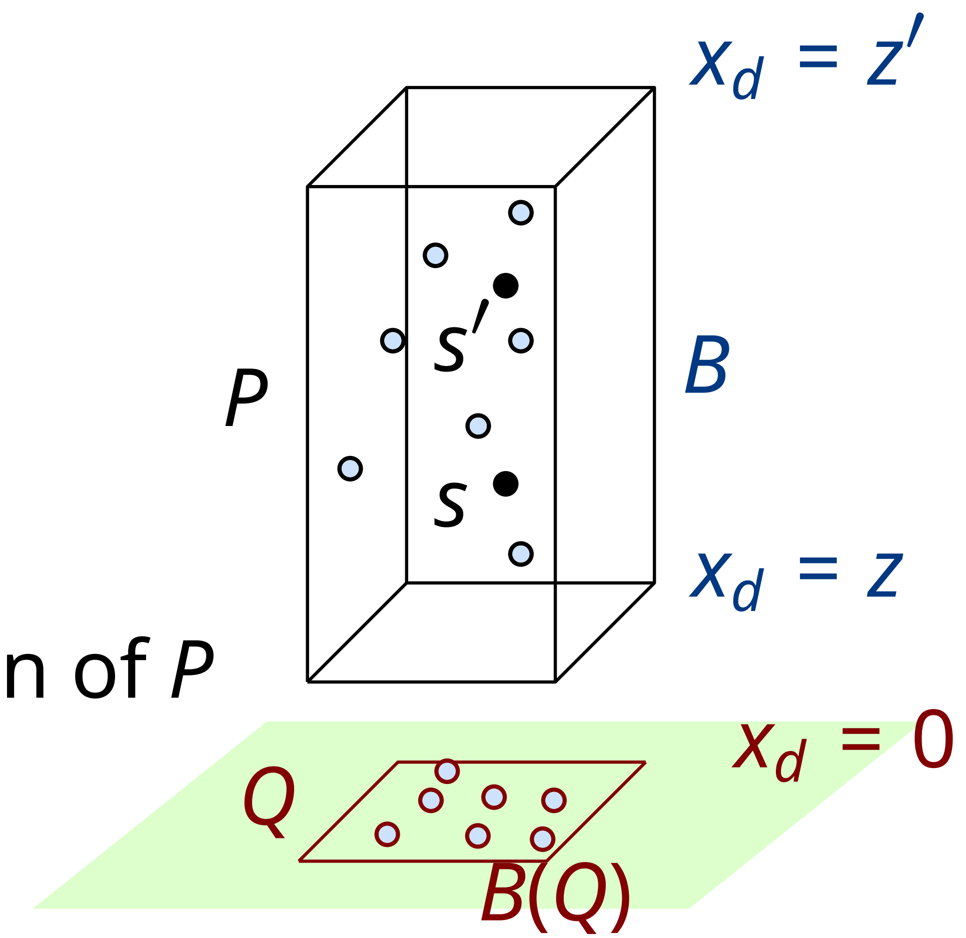


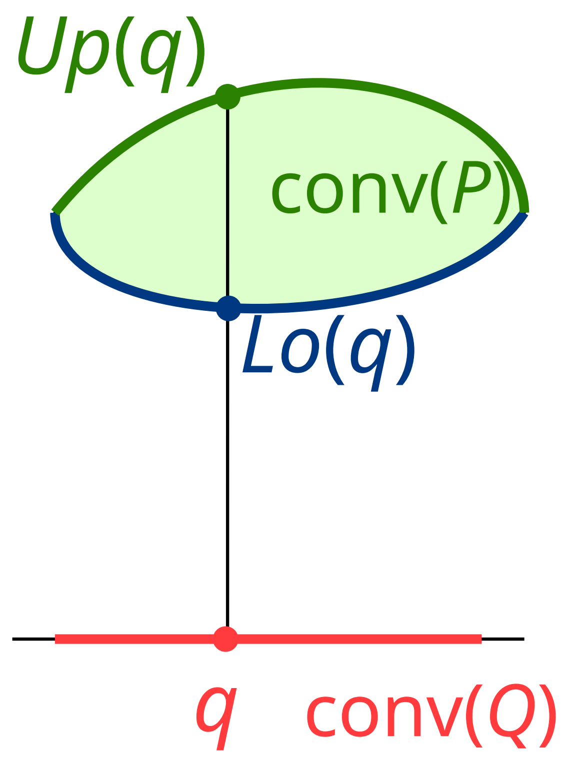
$$B := B(Q) \times [z, z']$$

Still need: $\mathrm{Vol}_d(\mathrm{conv}(P)) \geq \mathrm{Vol}_d(B)/(2^d d!)$

Volume bound

Upper hull $\mathrm{conv}^{\uparrow}(P)$ as function: $Up : \mathrm{conv}(Q) \rightarrow \mathbb{R}^d$ is concave

Lower hull $\mathrm{conv}^{\downarrow}(P)$ as function: $Lo : \mathrm{conv}(Q) \rightarrow \mathbb{R}^d$ is convex



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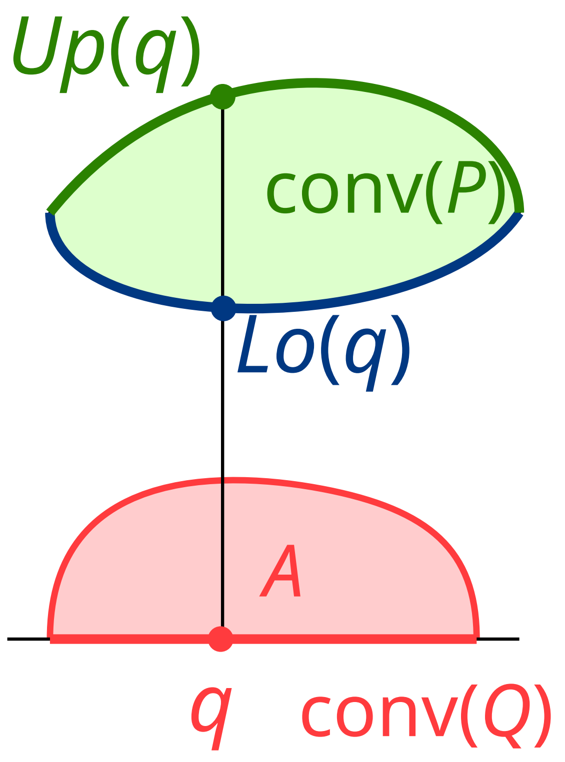
$\Rightarrow A := \bigcup_{q \in \text{conv}(Q)} [0, Up(q) - Lo(q)]$ is convex



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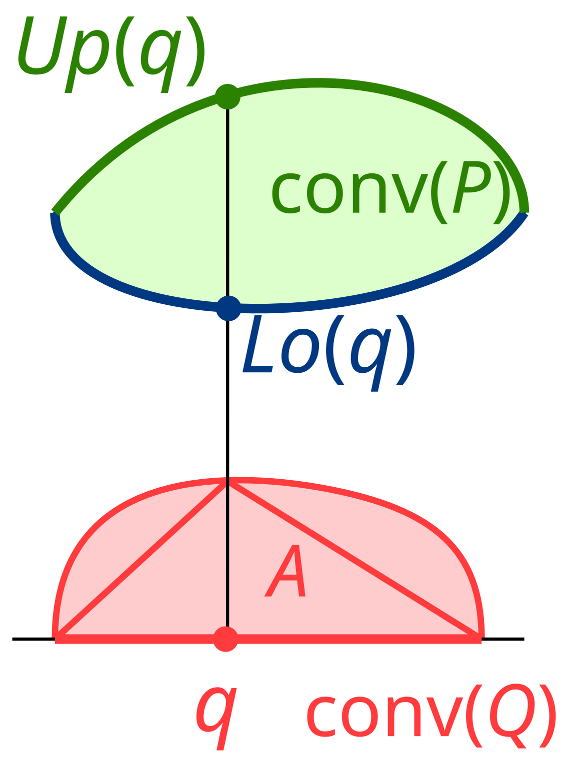
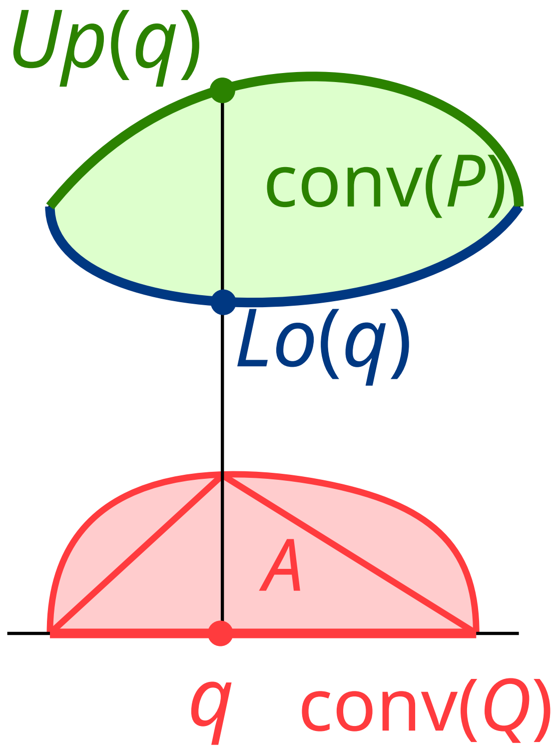


At $\pi(s)$, height of A is at least $|ss'|$.

A contains *pyramid* with base $\text{conv}(Q)$ and pole length $\geq |ss'|$.

Volume bound

$$
\begin{aligned}
\mathrm{Vol}_d(B) &\geq \mathrm{Vol}_d(B_{opt}) \\
&\geq \mathrm{Vol}_d(\mathrm{conv}(P)) = \mathrm{Vol}_d(A) \\
&\geq \mathrm{Vol}(pyramid) \\
&\geq \frac{\mathrm{Vol}_{d-1}(\mathrm{conv}(Q))|ss'|}{d} \\
&\geq \frac{\mathrm{Vol}_{d-1}\left(B(Q)/(2^{d-1}(d-1)!)\right)2|ss'|}{2d} \\
&\geq \frac{\mathrm{Vol}_{d-1}(B(Q))|zz'|}{2^d d!} \\
&= \frac{\mathrm{Vol}_d(B)}{2^d d!}
\end{aligned}
$$



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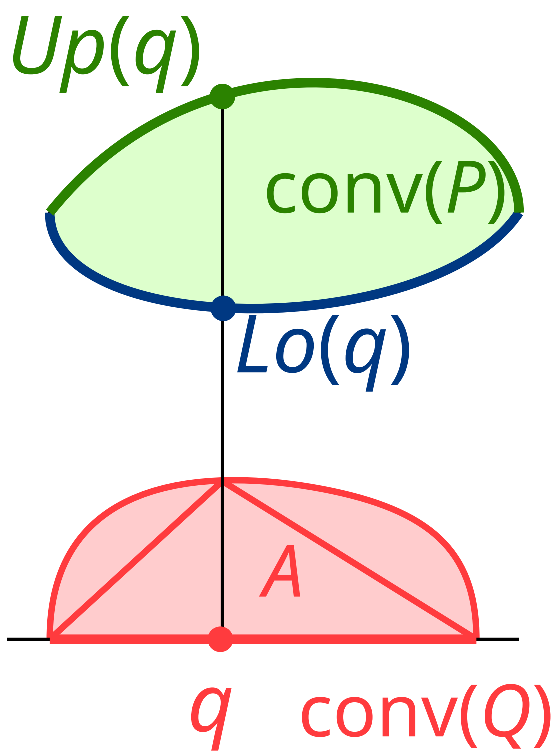

Running time: $T(n, d) = O(nd) + T(n, d-1) = O(nd^2)$.

Summary

Coreset: small (sub-)set capturing the relevant geometry

slow algorithm + coreset = fast approximation algorithm

a coreset is constructed for specific geometric optimization problem

Coreset for directional width:

construction using grids (+ bounding box)

solves various other problems too: min-volume bounding box, min-enclosing ball, diameter, ...