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Planarity Testing and Optimal Edge Insertion with Embedding Constraints

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Abstract. There is a variety of algorithms for testing if a graph is planar and for embedding a planar graph in the plane. However, many practical applications demand additional restrictions on an admissible embedding. In particular, constraints on the permitted (clockwise) order of the edges around a vertex, like so-called *side constraints*, abound. In this paper, we introduce a set of hierarchical embedding constraints that also comprises side constraints. We present linear time algorithms for testing if a graph is *ec-planar*, i.e., admits a planar embedding satisfying the given embedding constraints, as well as for computing such an embedding. Moreover, we characterize the set of all possible ec-planar embeddings using BC- and SPQR-trees.
We also consider the *optimal edge insertion problem* under the additional restrictions imposed by embedding constraints. The efficient algorithm [16] for solving this problem in the unconstrained case has already proven to be a valuable tool for crossing minimization; see [15]. We show that the optimal edge insertion problem subject to embedding constraints can still be solved in linear time.

1 Introduction

In many application domains information visualization is based on graph representations. Examples include software engineering, data bases, business process modeling, VLSI-design, and bioinformatics. The computation of concise graph layouts by automatic layout systems facilitates the readability and immediate understanding of the displayed information. These layout systems need to take into account application specific as well as user-defined layout rules in addition to the aesthetic criteria they try to optimize. In database diagrams, for example, links between attributes should enter the tables only at the left or right side of the corresponding attributes, the placement of reactants in chemical reactions or biological pathways should reflect their role within the displayed reactions, and in UML class diagrams, generalization edges should leave a class object at the top and enter a base class object at the bottom. Many of these layout rules impose restrictions on the admissible embeddings for a drawing. Even more important is the possibility to use drawing restrictions in order to express the user's preferences and to guide the layout phase. A general survey of constraints in graph drawing algorithms is given in [21].

In this paper, we consider restrictions on the allowed order of incident edges around a vertex, e.g., to specify groups of edges that have to appear consecutively around the vertex or that have a fixed clockwise order in any admissible embedding. Such constraints occur, e.g., in form of *side constraints*, where incident edges are assigned to the four sides of a rectangular vertex, or *port constraints* where edges have prescribed attachment points at a vertex. In particular, we introduce three types of constraints which may be

arbitrarily nested: *grouping*, *oriented* (prescribed clockwise order), and *mirror* constraints (prescribed reversible order). We call a planar embedding that fulfills the given set of constraints an *ec-planar* embedding.

Even though constraint handling is an important issue because of its relevance in practical applications, e.g., in interactive graph drawing (see, e.g., [2, 20, 6, 5]), there is only few previous work concerning constraints on the admissible embeddings of a graph. Di Battista et al. [8] consider embedding constraints that appear in database schemas, where table attributes are arranged from top to bottom within a rectangular vertex representing a table, and links that connect attributes may attach at the left or right hand side of these attributes. The integer linear programming approach in [12] considers side constraints in the shape computation phase of orthogonal graph drawing. Dornheim [11] studies the problem of computing embeddings satisfying topological constraints that consist of a cycle together with two sets of edges that have to be embedded inside or outside the cycle, respectively. On the other hand, linear time algorithms for planarity testing and embedding are long since known; see [17, 3, 7, 19, 4].

Our contribution is a linear time algorithm for testing if a graph with a set of embedding constraints is ec-planar. The main challenge is to incorporate oriented constraints, where a given clockwise order of (groups of) incident edges needs to be satisfied. Furthermore, we characterize all possible ec-planar embeddings using BC- and SPQR-trees, which also yields a linear time algorithm for computing an ec-planar embedding.

An important optimization goal for the computation of graph layouts is the minimization of crossings. The problem of minimizing the number of crossings in a drawing is NP-hard [13] and no practically efficient method exists so far. In practice, the problem is attacked via the *planarization* approach which first deletes a number of edges until the remaining graph is planar and then carefully reinserts them (iteratively) so that the number of crossings is minimized, see for example [15]. The *optimal edge insertion problem* asks for inserting an edge $e = (v, w)$ into a planar graph so that all crossings involve e and their number is minimized. Alternatively, the problem can be stated as finding an embedding of a planar graph G where the given edge can be inserted with the minimum number of crossings. Recently, the problem has been solved in linear time using the SPQR-tree data structure [16]. The algorithm essentially computes a shortest path Ψ between those nodes in the SPQR-tree $\mathcal{T}$ of G whose skeletons contain v and w, respectively. The optimal insertion path is constructed by simply concatenating locally optimal insertion paths of the tree nodes on Ψ.

However, if embedding constraints have to be observed, i.e., restrictions on the order of the edges around the vertices of G are given, locally optimal solutions need not lead to globally optimal solutions and the greedy approach cannot be applied anymore. The best local solution now depends on the decisions for other parts of the edge insertion path. Our contribution is a new linear time algorithm to solve the *ec-constrained optimal edge insertion problem*: Given an ec-planar graph G with an additional edge e and set of embedding constraints C for the graph $G + e$, our algorithm computes an *ec-planar* embedding of G together with a crossing minimal edge insertion path for e that observes C.

This paper is organized as follows. After recalling some known results on planar embeddings in Sect. 2, Sect. 3 formally defines the embedding constraints considered in this paper. The first part of the ec-planarity test consists of transforming the input graph into an *ec-expansion* which is described in Sect. 4; the characterization of ec-planar embeddings and the ec-planarity test itself is then presented in Sect. 5. Sect. 6 covers the

linear time algorithm for solving the ec-constrained edge insertion problem. Finally, we conclude the paper with remarks on open problems.

2 Preliminaries

For basic graph terminology, we refer the reader to [10]. A *combinatorial embedding* of a planar graph G is defined as a clockwise ordering of the incident edges for each vertex with respect to a crossing-free drawing of G in the plane. A *planar embedding* is a combinatorial embedding together with a fixed *external face*.

A *block* is a maximal 2-connected subgraph. The relationship between blocks and cut vertices is given by the *block-cutvertex tree*, or *BC-tree* for short. The *block-vertex tree* $\mathcal{B}$ of a connected graph G represents the relation between the blocks and vertices of G. It contains a B-node for each block of G and a V-node for each vertex of G; a V-node v and a B-node B are connected by an edge iff $v \in B$. The *representative* of a vertex v of G in block B is either v itself if $v \in B$, or the first vertex on the unique path from B to v in $\mathcal{B}$. If G is 2-connected, its *SPQR-tree* $\mathcal{T}$ represents the decomposition of G into its 3-connected components comprising serial, parallel, and 3-connected structures; see [22, 9] for a formal definition. The respective structure is given by a skeleton graph associated with each tree node, which is either a cycle (S-node), a bundle of parallel edges (P-node), or a 3-connected simple graph (R-node). We denote with $skeleton(\mu)$ the skeleton graph associated with node μ. In addition, Q-nodes serve as representatives for the edges of G. For each vertex v of G, the nodes in $\mathcal{T}$ whose skeletons contain v are called the *allocation nodes* of v.

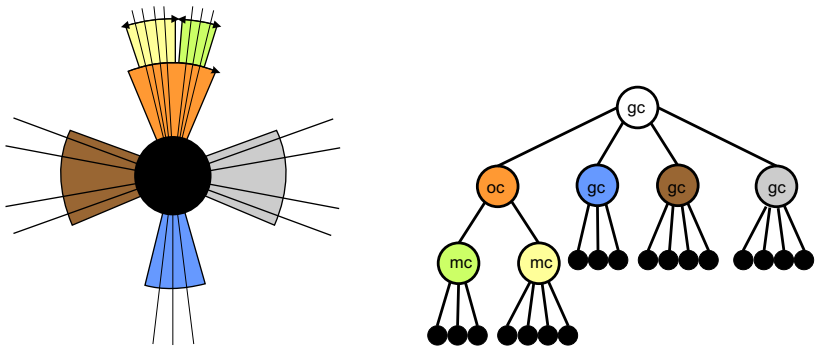
If G is 2-connected and planar, its SPQR-tree $\mathcal{T}$ represents all combinatorial embeddings of G. In particular, a combinatorial embedding of G uniquely defines a combinatorial embedding of each skeleton in $\mathcal{T}$, and fixing the combinatorial embedding of each skeleton uniquely defines a combinatorial embedding of G.

3 Embedding constraints

Let $G = (V, E)$ be a graph. An embedding constraint specifies the admissible clockwise order of the edges incident to a vertex in a combinatorial embedding of G. In this paper, we consider the case where a vertex has at most one embedding constraint and either all or none of the edges incident to a vertex are subject to embedding constraints.

An *embedding constraint* at a vertex $v \in V$ is a rooted, ordered tree T_v such that its leaves are exactly the edges incident to v. The inner nodes of T_v, also called *constraint-nodes* or *c-nodes* for short, are of three types: *oc-nodes* (oriented constraint-nodes), *mc-nodes* (mirror constraint-nodes), and *gc-nodes* (grouping constraint-nodes). Since T_v is an ordered tree, it imposes an order on its leaves and thus on the edges incident to v. We consider this order as a cyclic order and represent all *admissible* cyclic, clockwise orders of the edges incident to v by defining, how the order of the children of c-nodes in T_v can be changed:

gc-node: The order of children may be arbitrarily permuted.
mc-node: The order of children may be reversed.
oc-node: The order of children is fixed.



(a) Partitioning of edges. (b) Constraint tree.

Fig. 1: The hierarchical partitioning of edges imposed by an embedding constraint (a) and the corresponding constraint tree (b).

Fig. 1 shows an example for an embedding constraint. A c-node with a single child is obviously redundant, therefore we demand that each c-node has at least two children. While gc- and mc-nodes alone resemble the concept of PQ-trees [3], the additional concept of oc-nodes is necessary to model constraints that arise in many practical applications, and that complicate planarity testing.

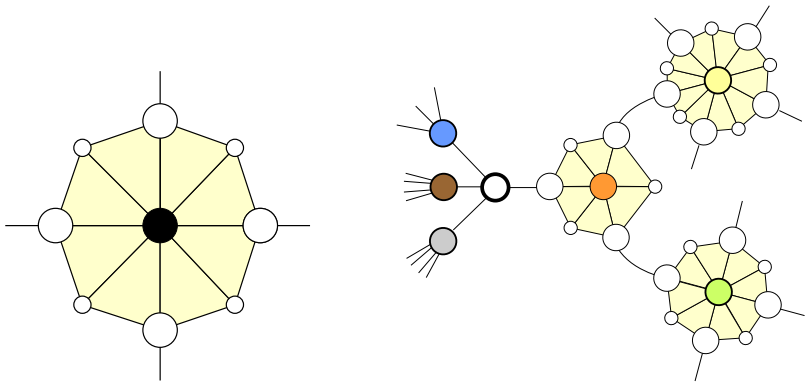
Let C be a set of embedding constraints at distinct vertices of G. A combinatorial embedding Γ of G *observes* the embedding constraints in C, if for each embedding constraint $T_v \in C$, the cyclic clockwise order of the edges around v in Γ is admissible with respect to T_v. A planar embedding observing the embedding constraints in C is an *ec-planar embedding* with respect to C, and (G, C) is *ec-planar*, if there exists an ec-planar embedding of G with respect to C.

4 ec-Expansion

A basic building block of the ec-planarity test is a structural transformation applied to a given graph G with embedding constraints C. For each embedding constraint T_v at vertex v, this transformation expands v according to the structure of T_v. We call the resulting graph the *ec-expansion* $E(G, C)$ of G with respect to C. The details of this transformation are given below.

4.1 Construction of the ec-Expansion

The *ec-expansion* $E(G, C)$ of G with respect to C is constructed as follows. Let $T_v \in C$ be an embedding constraint and T'_v the subgraph obtained from T_v by omitting its leaves. Recall that the leaves of T_v are exactly the edges incident to v. We replace v in G by the tree T'_v and connect the edges incident with v with the parents of the corresponding leaves. This transformation introduces a vertex in G for every c-node in T_v. Each vertex u corresponding to an oc- or mc-node is further replaced by a *wheel gadget* which is a wheel graph with $2d$ spokes, were $e_1, \ldots, e_d$ are the edges incident to u. Then, the respective wheel gadget consists of a cycle $x_1, y_1, \ldots, x_d, y_d$ of length $2d$ and a vertex, called *hub*, incident to every vertex on the cycle; see Fig. 2(a). The vertex u is replaced by



(a) Wheel gadget. (b) Vertex expansion.

Fig. 2: Expansion gadgets: (a) a wheel gadget replacing a vertex with degree 4; (b) vertex expansion according to the constraint tree in Fig. 1(b) (thick hollow vertex is the root).

this wheel gadget, such that e_i is connected to x_i for $1 \leq i \leq d$. According to the type of the expanded c-node, we distinguish between *O-hubs* (oc-nodes) and *M-hubs* (mc-nodes). We refer to the edges introduced during the ec-expansion as *expansion edges*. Fig. 2(b) shows the expansion of a vertex according to the constraint tree shown in Fig 1(b).

The purpose of the wheel gadgets is to model the fixed order of the children of the corresponding c-node. Since a wheel gadget is a 3-connected graph, it admits only two combinatorial embeddings that are mirror images of each other. The order in which non-gadget edges are attached to the wheel cycle is either the order given by the corresponding c-node, or the reverse order. Every face adjacent to the hub is a triangle. We call these faces *inner wheel gadget faces*.

Lemma 1. *Let $G = (V, E)$ be a graph with embedding constraints C. Then, its ec-expansion $E(G, C)$ has size $O(|V| + |E|)$ and can be constructed in time $O(|V| + |E|)$.*

Proof. Consider an embedding constraint $T_v \in C$. Since the leaves of T_v are in one-to-one correspondence to the edges incident to v and each c-node has at least two children, the size of T_v is linear in $\deg(v)$. We replace each oc- and mc-node μ by a wheel gadget with $4\deg(\mu)$ edges. Thus, the expansion of vertex v creates $\mathrm{O}(\deg(v))$ edges, and the total number of additional edges in $E(G, C)$ is bounded by $\sum_{v \in V} \mathrm{O}(\deg(v)) = \mathrm{O}(|E|)$. Therefore, the size of the expansion graph is $\mathrm{O}(|V|+|E|)$, and the expansion can obviously be computed in $\mathrm{O}(|E(G, C)|) = \mathrm{O}(|V| + |E|)$ time. □

4.2 ec-Expansion and ec-Planar Embeddings

In this section we discuss the relationship between planar embeddings of the ec-expansion $E(G, C)$ and ec-planar embeddings of (G, C). Though the ec-expansion serves as a tool for modeling the embedding constraints in C, a planar embedding of $E(G, C)$ needs to fulfill certain conditions in order to induce an ec-planar embedding of G with respect to C. We call a planar embedding Γ of $E(G, C)$ *ec-planar* if

1. the external face of Γ does not contain a hub;

2. every face incident to a hub is a triangle consisting solely of edges of the corresponding wheel gadget; and
3. each O-hub h is *oriented correctly*, i.e., the cyclic, clockwise order of the edges around h in Γ corresponds to the order specified by the corresponding oc-node.

Let Γ be an ec-planar embedding of $E(G, C)$. Then, we obtain an ec-planar embedding of (G, C) as follows. For each vertex v with corresponding embedding constraint in C, there is a connected subgraph G_v in $E(G, C)$ resulting from expanding v. Let $\bar{G}_v \subset E(G, C)$ be the rest of the graph, i.e., the graph induced by the vertices not contained in G_v. The conditions above assure that the planar embedding Γ_v of G_v induced by Γ is such that $\bar{G}_v$ lies in the external face of Γ_v. The edges that connect G_v to $\bar{G}_v$ correspond to the edges incident to v in G. Their cyclic clockwise order around G_v is admissible with respect to T_v, since the wheel gadgets fix the order of the edges specified by oc- and mc-nodes, and O-hubs are oriented correctly. We shrink G_v to a single vertex by contracting all edges in G_v while preserving the embedding, thus resulting in an admissible order of the edges around v.

If we have an ec-planar embedding of (G, C), then the edges around each vertex v are ordered such that the constraints in T_v are fulfilled. It is easy to see that we can replace each such vertex v by the expansion graph corresponding to T_v in such a way that we obtain an ec-planar embedding of $E(G, C)$. Thus, we get the following result:

Lemma 2. *Let G be a graph with embedding constraints C. Then, (G, C) is ec-planar if and only if $E(G, C)$ is ec-planar. Moreover, every ec-planar embedding of $E(G, C)$ induces an ec-planar embedding of (G, C).*

5 ec-Planarity Testing

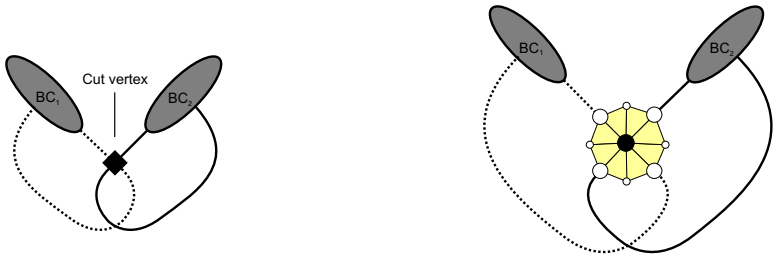
It is well-known that planarity testing can be reduced to 2-connected graphs, i.e., it is sufficient to test the blocks of a graph independently. However, adding embedding constraints complicates this task. Let G be a graph with embedding constraints C. Consider a cut vertex c in G that connects two blocks BC_1 and BC_2 via the edge sets s_1 and s_2, respectively; see Fig. 3(a). If these edge sets are subject to embedding constraints that force the edges in s_1 and s_2 to be intermixed as in Fig. 3(a), then the given graph is not ec-planar even if its blocks are ec-planar. We solve this problem by first applying the ec-expansion to the graph. This replaces the cut vertex c by a wheel gadget so that c does not separate BC_1 and BC_2 anymore; see Fig. 3(b).

By Lemma 2, we know that it is sufficient to test the ec-expansion $E(G, C)$ for ec-planarity. In contrast to the graph G itself, the following lemma shows that we can test the blocks of $E(G, C)$ separately.

Lemma 3. *$E(G, C)$ is ec-planar iff every block of $E(G, C)$ is ec-planar.*

Proof. If $E(G, C)$ is ec-planar, then there is an ec-planar embedding of $E(G, C)$, and this embedding implies an ec-planar embedding for each block of $E(G, C)$.

Suppose now that each block of $E(G, C)$ is ec-planar. Consider a wheel gadget $\mathscr{G}$ in $E(G, C)$. Since $\mathscr{G}$ is 3-connected, $\mathscr{G}$ is completely contained in a single block B of $E(G, C)$. For each edge $(u, v) \in \mathscr{G}$, the pair $\{u, v\}$ is not a separation pair in B by construction, hence every inner wheel face of $\mathscr{G}$ is also a face in every planar embedding of B. Moreover, the hub of $\mathscr{G}$ is not a cut vertex of $E(G, C)$, since all its incident edges are in B.



(a) (b)

Fig. 3: A crossing is needed between edges of two 2-connected components due to embedding constraints (a). The expansion using a wheel gadget merges the two components (b).

We construct an ec-planar embedding of $E(G, C)$ as follows. We start with an arbitrary block B of $E(G, C)$. Let Π be an ec-planar embedding of B. In particular, the external face of Π is not an inner wheel face of a wheel gadget. We add the remaining blocks successively to Π. Let B' be another block of $E(G, C)$ that shares a vertex c with B, and let Π' be an ec-embedding of B'. We pick faces $f \in \Pi$ and $f' \in \Pi'$ that are adjacent to c and not inner wheel faces of a wheel gadget. This is possible, since the only vertices adjacent solely to inner wheel faces are the O- and M-hubs. Then, we insert Π' with f' as external face into the face f of Π. This results in an ec-planar embedding of $B \cup B'$. We can add the remaining blocks (if any) in the same way, resulting in an ec-planar embedding of $E(G, C)$. □

If we can characterize all ec-planar embeddings of the blocks of $E(G, C)$, the construction in the proof of Lemma 3 also shows, how to enumerate all ec-planar embeddings of $E(G, C)$ by traversing its BC-tree. In the following, we devise such a characterization. Let B be a block of $E(G, C)$ and $\mathcal{T}$ its SPQR-tree.

Observation 1. *Every wheel gadget $\mathscr{G}$ is completely contained within the skeleton of an R-node. In particular, the hub of $\mathscr{G}$ occurs only in the skeleton of a single R-node.*

Proof. $\mathscr{G}$ is 3-connected, and for each edge $(u, v) \in \mathscr{G}$, the pair $\{u, v\}$ is not a separation pair in B by construction. Therefore, all edges of $\mathscr{G}$ occur in the same skeleton graph, which must be the skeleton of an R-node μ. The hub h of $\mathscr{G}$ is only incident to edges of $\mathscr{G}$ and no other edge of B, hence h occurs only in $skeleton(\mu)$. □

If B is planar, then the skeleton of an R-node is a 3-connected planar graph, thus having exactly two planar embeddings which are mirror images of each other. We call two O-hubs contained in the same skeleton S *conflicting* if none of the two planar embeddings of S orients both O-hubs correctly. The following theorem gives us an easy to check condition for ec-planarity and characterizes all possible ec-planar embeddings:

Theorem 1. *Let G be a graph with embedding constraints C. Let B be a block of $E(G, C)$ and $\mathcal{T}$ its SPQR-tree. Then, the following holds:*

1. *B is ec-planar iff B is planar and no skeleton of an R-node of $\mathcal{T}$ contains conflicting O-hubs.*
2. *If B is ec-planar, then the embeddings of the skeletons of $\mathcal{T}$ induce an ec-planar embedding of B iff each O-hub in the skeleton of an R-node is oriented correctly.*

```
1: function IsEcPlanar(Graph G, Constraints C) : bool
2:     Construct ec-expansion E of (G, C).
3:     if E is not planar then return false
4:     for each block B of E do
5:         Construct SPQR-tree T of B.
6:         for each R-node μ ∈ T do
7:             if skeleton(μ) contains two conflicting O-hubs then
8:                 return false
9:             end if
10:        end for
11:    end for
12:    return true
13: end function
```

Algorithm 1: Ec-planarity testing.

Proof. If B admits an ec-planar embedding, then this embedding induces embeddings of the skeletons of $\mathcal{T}$ such that every O-hub in the skeleton of an R-node is oriented correctly. In particular, no R-node skeleton contains conflicting O-hubs.

Suppose now that B is planar and no R-node skeleton contains conflicting O-hubs. For each R-node skeleton containing at least one O-hub, we can chose planar embeddings such that all O-hubs are oriented correctly within the skeletons. We have to show that the embeddings of the skeletons induce an ec-planar embedding of B, even if we chose arbitrary embeddings for the remaining skeletons. This holds, since every such embedding Π has the property that each O-hub is oriented correctly because wheel gadgets are completely contained within R-node skeletons by Observation 1, and inner wheel faces are preserved. We can pick any face of Π as external face which is not an inner wheel face (such a face always exists) and obtain an ec-planar embedding of B. □

Function IsEcPlanar depicted in Algorithm 1 applies Theorem 1 and devises a linear time ec-planarity test, which can easily be extended so that it computes an ec-planar embedding as well.

Theorem 2. *Let $G = (V, E)$ be a graph with embedding constraints C. Then, algorithm* IsEcPlanar *tests (G, C) for ec-planarity in time $O(|V| + |E|)$. Moreover, if (G, C) is ec-planar, an ec-planar embedding of (G, C) can also be computed in time $O(|V| + |E|)$.*

Proof. By Lemma 2 and 3, it is sufficient to test every block of $E(G, C)$ for ec-planarity. Hence, the correctness of Algorithm 1 follows from Theorem 1.

Constructing the ec-expansion (Lemma 1) and testing planarity [17] can be done in linear time. For each block B of $E(G, C)$, we construct its SPQR-tree, which requires linear time in the size of B; see [14]. The check for conflicting O-hubs is easy to implement: For each R-node skeleton S, we compute a planar embedding of S. If this embedding contains both correctly as well as not correctly oriented O-hubs, then there is a conflict, otherwise not. Since the total size of skeleton graphs is linear in the size of B and a planar embedding can be found in linear time (see, e.g., [7]), we need linear running time for each block. Hence, the total running time is linear in the size of $E(G, C)$ which is $O(|V| + |E|)$ by Lemma 1.

In order to find an ec-planar embedding of G, we just have to compute embeddings of the skeleton graphs for each block as described in Theorem 1 and combine the embeddings as described in the proof of Lemma 3. □

6 ec-Edge Insertion

6.1 ec-Edge Insertion Paths and ec-Traversing Costs

We first generalize the terms insertion path and traversing costs introduced in [16]. Intuitively, the edges in an insertion path are the edges we need to cross when inserting an edge (x, y) into an embedding. Let $G + (x, y)$ be a graph with embedding constraints C. An *ec-edge insertion path* for x, y in an ec-planar embedding Π of G is a sequence of edges $e_1, \ldots, e_k$ of G satisfying the following conditions:

1. There is a face $f_x \in \Pi$ with $x, e_1 \in f_x$, a face $f_y \in \Pi$ with $e_k, y \in f_y$, and faces $f_i \in \Pi$ with $e_i, e_{i+1} \in f_i$ for $1 \leq i < k$.
2. The edge order around x and y is admissible with respect to C if (x, y) leaves x via face f_x and enters y via face f_y.

Finding a shortest ec-insertion path in a fixed embedding Π is easy: We only need to identify the set of faces F_x incident to x where the insertion path may start, and F_y incident to y where it may end, and then find a shortest path in the dual graph of Π connecting a face in F_x with a face in F_y.

We are interested in the shortest possible ec-insertion path among all ec-planar embeddings of G, which we also call an *optimal ec-insertion path* in G. In particular, we need to identify the required ec-planar embedding of G. In order to represent all ec-planar embeddings of G, we apply Lemma 2 and use its ec-expansion instead. More precisely, we use the subgraph $K = E(G+(x, y), C) \setminus e$, where $e = (v, w)$ is the edge of $E(G+(x, y), C)$ connecting the expansion of x with the expansion of y. An ec-insertion path in an ec-planar embedding of K is defined as before with the only difference that we replace the second condition with

$2'$. $e_1, \ldots, e_k$ contains no expansion edge of K.

It is easy to see that we can also use this definition for a subgraph B of K and two distinct vertices of B that are not hubs.

We adapt the notion of traversing costs defined in [16] to ec-planarity. Let e be a skeleton edge, and let Π be an arbitrary ec-embedding of the graph $\mathit{expansion}^+(e)$ (which is the expansion graph of e plus the edge e) with dual graph Π^*, in which all edges corresponding to gadget edges have length ∞ and the other edges have length 1. Let f_1 and f_2 be the two faces in Π separated by e. We denote with $P(\Pi^*, e)$ the length of the shortest path in Π^* that connects f_1 and f_2 and does not use the dual edge of e. Hence, we have $P(\Pi, e) \in \mathbb{N} \cup \{\infty\}$.

The following lemma follows analogously to the result shown in [16].

Lemma 4. *Let μ be a node in $\mathcal{T}$ and let e be an edge in skeleton(μ). Then, the length of the path $P(\Pi^*, e)$ is independent of the ec-embedding Π of $\mathit{expansion}^+(e)$.*

Proof. Let m be the number of edges in $G_e := \mathit{expansion}^+(e)$ and G'_e be the graph obtained from G_e by replacing each gadget edge with $m + 1$ parallel edges. Then, each

embedding Π of G_e corresponds to an embedding Π' of G'_e, and the length of the path $P(\Pi, e)$ is ∞ if and only if the corresponding path in Π' is longer than m. Applying Lemma 1 in [16] and observing that the ec-embeddings of G_e are a non-empty subset of the embeddings of G_e yields the lemma. □

Thus, we define the *ec-traversing costs* $c(e)$ of a skeleton edge e as the length of the path $P(\Pi^*, e)$ for an arbitrary ec-embedding Π of $expansion^+(e)$.

6.2 The Algorithm for 2-connected Graphs

The hard part is to find an ec-insertion path in a block B of K. Our task is to compute an optimal ec-insertion path between two nodes v, w of B. The function OptimalEcBlockInserter shown in Algorithm 2 and 3 solves this problem. It is called with a block B of an ec-planar ec-expansion and two distinct vertices v and w of B. Since we assume that B contains all gadget edges, we do not need to pass further constraint information for the edge (v, w). In particular, using any insertion path in any ec-planar embedding of B that connects v and w and does not cross a gadget edge yields an ec-embedded planarization of $B \cup (v, w)$. Hence, we look for an ec-embedding of B that allows the insertion of the edge (v, w) with the minimum number of crossings.

It starts by computing the SPQR-tree $\mathcal{T}$ of B and embeds the skeletons such that they imply an ec-embedding of B, i.e, the R-node skeletons are embedded correctly. Then, the shortest path $\Upsilon := \mu_1, \ldots, \mu_k$ between an allocation node of v and of w is identified. In order to achieve a consistent orientation, we root $\mathcal{T}$ such that Υ is a descending path in the tree, i.e., μ_i is the parent of μ_{i-1} for $i = 2, \ldots, k$. Note that the rooting of the SPQR-tree implies a direction of the skeleton edges: the edges in a skeleton with reference edge $e_r = (s, t)$ are directed such that the skeleton is a planar st-graph; see, e.g., [9]. This direction is necessary in order to identify the left and the right face of an edge.

The algorithm traverses the path Υ from μ_1 to μ_{k-1} and iteratively computes the lengths of the shortest ec-insertion paths that start from v and leave the pertinent graph P_i of μ_i to the left or to the right, respectively, where all ec-embeddings of P_i are considered. Here, left and right refer to the direction of the reference edge of μ_i. These lengths are maintained in the variables λ_ℓ and λ_r. Finally, when node μ_k is considered, this information is used to determine a shortest insertion path ending at w.

For each node μ_i, the following information is computed:

- ϕ^i_ℓ (resp. ϕ^i_r) indicates if the shortest ec-insertion path leaving P_i to the left (right) uses the shortest ec-insertion path that leaves P_{i-1} to the left (in this case the value is ℓ) or to the right (the value is r).
- Δ^i_ℓ (resp. Δ^i_r) is the subpath that is appended to the path leaving P_{i-1} when leaving P_i to the left (right).

These values are solely used for the purpose of creating the optimal ec-insertion path at the end of the procedure. If $s \in \{\ell, r\}$ denotes a side, we denote with $\bar{s}$ the other side, i.e., $\bar{\ell} = r$ and vice versa.

The body of the for-loop starts by expanding all edges of the skeleton S_i of μ_i except for edges representing v or w. The resulting graph is called G_i. If $1 < i < k$, then G_i will contain two virtual edges e_v (representing v) and e_w (representing w). Note that we obtain P_i (plus reference edge) by replacing e_v with P_{i-1}.

We distinguish according to the type of μ_i. If μ_i is a P-node, then the optimal ec-insertion path leaving P_{i-1} to the left (right) is also an optimal ec-insertion path leaving

procedure OptimalEcBlockInserter(Block B of K, *vertex* v, *vertex* w)

Construct SPQR-tree $\mathcal{T}$ of B such that the embeddings of the skeletons imply a feasible embedding of B.

Find the shortest path $\mu_1, \ldots, \mu_k$ in $\mathcal{T}$ between an allocation node μ_1 of v and μ_k of w.
Root $\mathcal{T}$ such that μ_k becomes the parent of μ_{k-1} (if $k > 1$).

$\lambda_\ell := \lambda_r := 0$ ▷ *length of shortest insertion path leaving to the left/ right*

for $i = 1, \ldots, k$ **do**

let $S_i = skeleton(\mu_i)$

let G_i be the graph obtained from S_i by replacing each edge not representing v or w with its expansion graph, and let Π_i be the embedding of G_i induced by the embeddings of the skeletons of $\mathcal{T}$.

▷ $\phi^i_{l/r}$ *indicates which insertion path of* μ_{i-1} *is chosen when leaving left/right*
▷ $\Delta^i_{l/r}$ *denotes the subpath within* S_i *when leaving left/right*

if μ_i is a P-node **then**
$(\phi^i_\ell, \Delta^i_\ell) := (\ell, \epsilon)$; $(\phi^i_r, \Delta^i_r) := (r, \epsilon)$ ▷ *no crossings required*

else ▷ *S- or R-node*

if $i = 1$ **then**
$L_v := R_v :=$ the set of adjacent faces of the copy of v in S_i

else
let e_v be the representative of v in S_i
$L_v :=$ { the left face of e_v}
$R_v :=$ { the right face of e_v}

end if

if $i = k$ **then**
$L_w := R_w :=$ the set of adjacent faces of the copy of w in S_i

else
let e_w be the representative of w in S_i
$L_w :=$ { the left face of e_w}
$R_w :=$ { the right face of e_w}

end if

▷ *Compute shortest ec-insertion paths (from l/r to l/r) within* G_i.
▷ *Note:* $p_{\ell r} = p_{\ell\ell}$ *and* $p_{rr} = p_{r\ell}$ *if* $i \in \{1, k\}$.
$p_{\ell r} :=$ ShortestEcInsPath(Π_i, L_v, L_w)
$p_{\ell\ell} :=$ ShortestEcInsPath(Π_i, L_v, R_w)
$p_{rr} :=$ ShortestEcInsPath(Π_i, R_v, L_w)
$p_{r\ell} :=$ ShortestEcInsPath(Π_i, R_v, R_w) ▷ *continued on next page...*

Algorithm 2: Computation of an optimal ec-insertion path (2-connected case).

```
            ▷ Collect possible solutions.
            Λ_ℓ := { (λ_ℓ + |p_ℓℓ|, ℓ, p_ℓℓ), (λ_r + |p_rℓ|, r, p_rℓ) }
            Λ_r := { (λ_ℓ + |p_ℓr|, ℓ, p_ℓr), (λ_r + |p_rr|, r, p_rr) }
            if μ_i is an R-node that can be mirrored then
                Λ_ℓ := Λ_ℓ ∪ { (λ_ℓ + |p_rr|, ℓ, p*_rr), (λ_r + |p_ℓr|, r, p*_ℓr) }
                Λ_r := Λ_r ∪ { (λ_ℓ + |p_rℓ|, ℓ, p*_rℓ), (λ_r + |p_ℓℓ|, r, p*_ℓℓ) }
            end if
            ▷ Pick best solution.
            (λ_ℓ, φ^i_ℓ, Δ^i_ℓ) := min_{1,3} Λ_ℓ
            (λ_r, φ^i_r, Δ^i_r) := min_{1,3} Λ_r
        end if
    end for
    ▷ Build final ec-insertion path. Note: λ_ℓ = λ_r always holds here!
    s_k := ℓ                                          ▷ Start with empty path.
    for i := k downto 1 do                            ▷ Collect path backward.
        p_i := Δ^i_{s_i};  s_{i−1} := φ^i_{s_i}
    end for
    return p_1 + ··· + p_k
end procedure
```

Algorithm 3: Procedure OptimalEcBlockInserter (part 2).

P_i to the left (right); we just need to permute the parallel edges in S_i such that e_v is the leftmost (rightmost) edge. Otherwise, we have four possibilities for extending an ec-insertion path leaving P_i. Such a path may start in a face left or right of e_v, and may end in a face left or right of e_w. In addition, we have to consider two special cases: if $i = 1$ then G_i contains v and the ec-insertion path may start in any face adjacent to v; if $i = k$ then G_i contains w and the ec-insertion path may end in any face adjacent to w. We compute the (at most) four possible shortest ec-insertion paths using the function ShortestEcInsPath(Π, F_s, F_t). Here Π is an ec-embedding of an ec-expansion, F_s are the faces where the insertion path may start, and F_t are the faces where it may end. The ec-insertion path is found using BFS in the dual graph of Π, where edges corresponding to gadget edges are removed (which means that it is forbidden to cross their primal counterparts). We call these shortest ec-insertion paths $p_{\ell\ell}, p_{\ell r}, p_{r\ell}, p_{rr}$, where $p_{\ell\ell}$ stands for the path starting in a face in L_v and ending in a face in R_w etc. We have two choices for a shortest ec-insertion path leaving P_i to the left if we consider only the given embedding of the skeleton of μ_i:

- We leave P_{i-1} to the left (or start at v if $i = 1$) and end in a face in R_w (e.g., we enter e_w from right). This path has length $\lambda_\ell + |p_{\ell\ell}|$.
- We leave P_{i-1} to the right (or start at v if $i = 1$) and end in a face in R_w (e.g., we enter e_w from left). This path has length $\lambda_r + |p_{r\ell}|$.

For the shortest ec-insertion path leaving P_i to the right, we have two similar cases. Further choices are possible if μ_i is an R-node that can be mirrored. We could mirror the embedding of S_i, expand the skeleton edges as before such that we obtain an embedding $\tilde{\Pi}_i$, and compute the four paths in $\tilde{\Pi}_i$ again. Notice that $\tilde{\Pi}_i$ is not simply the mirror image of Π_i. However, this is not necessary. We observe that, e.g., the path $\tilde{p}_{\ell\ell}$ is obtained

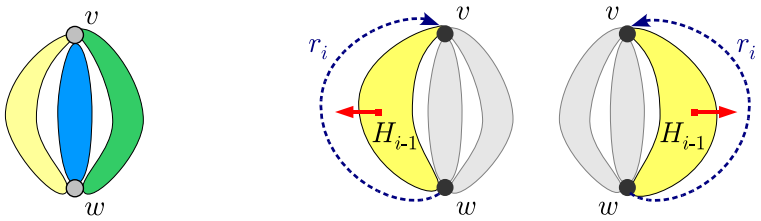


Fig. 4: Proof of Lemma 5; $k = 1$ and μ_1 is a P-node (left), and μ_i is a P-node (right).

from p_{rr} by reversing the subsequences of edges that have been created by expanding a common skeleton edge of S_i. We call this path p^*_{rr}. A similar argumentation holds for $\tilde{p}_{\ell r}, \tilde{p}_{r\ell}, \tilde{p}_{rr}$. It follows that we have at most four possible choices for leaving P_i to the left and to the right, respectively. Among all possible choices, we pick the shortest one.

After processing all nodes μ_i, it is easy to reconstruct the best ec-insertion path from v to w using $\phi^i_{\ell/r}$ and $\Delta^i_{\ell/r}$. Notice that $\lambda_\ell = \lambda_r$ holds at the end, since $L^k_w = R^k_w$.

6.3 Correctness and Optimality

Lemma 5. *There exists an ec-embedding Π of B such that $p_1 + \cdots + p_k$ is an ec-insertion path for v and w in B with respect to Π.*

Proof. Consider the path $\Upsilon = \mu_1, \ldots, \mu_k$ computed by the algorithm. By construction of Υ, the skeleton of μ_1 contains v, the skeleton of μ_k contains w, and, for each $j = 2, \ldots, k-1$, the skeleton of μ_j contains neither v nor w. Moreover, Υ does not contain a Q-node.

First, we prove the lemma for the case where Υ consists of a single node μ_1. In this case, the skeleton of μ_1 contains both v and w. We distinguish two cases according to the type of μ_1:

1. μ_1 **is a P-node.** Let Π be an arbitrary ec-embedding of B. Since v and w share a common face in Π, the empty path returned by the algorithm is an ec-insertion path for v and w in B with respect to Π; see Fig. 4 (left).
2. μ_1 **is an S- or an R-node.** In this case the graph G_1 constructed by the algorithm is the original block B, since all skeleton edges are expanded. Moreover, Π_1 is an ec-embedding of B, and $p_{\ell r} = p_{\ell\ell}$ and $p_{rr} = p_{r\ell}$ are ec-insertion paths in B with respect to Π_1. We do not need to consider the case where the embedding of the skeleton can be mirrored, since this will not yield a shorter path. Hence, p_1 is either $p_{\ell r}$ or p_{rr} and thus an ec-insertion path in B with respect to Π_1.

Assume now that $k > 1$. For $i = 1, \ldots, k$, we denote with H_i the pertinent graph of μ_i, with r_i the reference edge of μ_i in H_i, and, for $1 < i$, with e_i the edge in $\mathit{skeleton}(\mu_i)$ whose pertinent node is μ_{i-1}. Recall that $s_i \in \{\ell, r\}$ is the side of H_i where the computed insertion path shall leave. We show by induction over i that, for $1 \leq i < k$, there is an embedding Γ_i of H_i such that $p_1 + \cdots + p_i$ is an ec-insertion path leaving H_i at side s_i. The embeddings $\Gamma_1, \ldots, \Gamma_{k-1}$ are iteratively constructed during the proof. For our convenience, we denote with Γ_i^- the embedding of $H_i - r_i$ induced by Γ_i.

$i = 1$**.** Consider the different types for node μ_1:

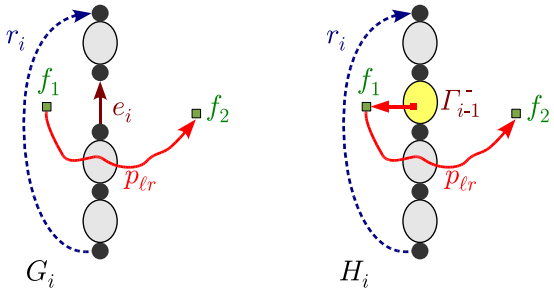


Fig. 5: Proof of Lemma 5; μ_i is an S-node, $L_v = R_w = \{f_1\}$, $R_v = L_w = \{f_2\}$.

1. μ_1 **is a P-node.** This case does not apply, since μ_2 is not an allocation node of v.
2. μ_1 **is an S- or an R-node.** In this case, $G_1 = H_1$, and p_1 is a path leaving either Π_1 or the mirror image of Π_1 to side s_1. Hence, we set Γ_1 to Π_1 or its mirror image, respectively.

$1 < i < k$**.** We distinguish again between the types of μ_i.

1. μ_i **is a P-node.** In this case, $p_i = \epsilon$, i.e., no further edges need to be crossed. The embedding Γ_i is obtained as follows. If $s_i = \ell$, we permute the edges in $\mathit{skeleton}(\mu_i)$ such that e_i is to the right of r_i; otherwise, we permute the edges such that e_i is to the left of r_i. Then, we replace e_i by Γ_{i-1}^-, and the remaining edges $e \neq r_i$ in $\mathit{skeleton}(\mu_i)$ by an arbitrary embedding of $\mathit{expansion}(e)$; see Fig. 4 (right).
2. μ_i **is an S- or an R-node.** In this case, p_i is either $p_{s_{i-1}s_i}$ or $p_{\bar{s}_{i-1}\bar{s}_i}$; the latter case corresponds to mirroring the embedding of $\mathit{skeleton}(\mu_i)$ before.
   We first restrict us to the case in which p_i is set to $p_{s_{i-1}s_i}$, i.e., an ec-insertion path in the embedding Π_i that starts in a face at side s_{i-1} of e_i and ends in a face at side $\bar{s}_i$ of edge r_i. We obtain Γ_i by replacing e_i by Γ_i^- in Π_i; see Fig. 5. Since the ec-insertion path $p_1 + \cdots + p_{i-1}$ leaves Γ_i^- to the side s_{i-1}, $p_1 + \cdots + p_i$ is an ec-insertion path leaving Γ_i to the side s_i.
   Finally, assume that $p_i = p_{\bar{s}_{i-1}\bar{s}_i}$. Let $\tilde{\Pi}_i$ be the embedding that we obtain by first mirroring the embedding of $\mathit{skeleton}(\mu_i)$ and then expanding and embedding each skeleton edge not representing v or w as before. We observe that p_i is an ec-insertion path in $\tilde{\Pi}_i$ that starts in a face at side s_{i-1} of e_i and ends in a face at side $\bar{s}_i$ of edge r_i; see Fig. 6. With the same argumentation as above, we obtain Γ_i by replacing e_i with Γ_{i-1}^- in $\tilde{\Pi}_i$.

To conclude the proof, we consider the node μ_k. We know that μ_k is either an S- or an R-node, and we may assume that $p_k = p_{s_{i-1}s_i}$, since $p_{\ell r} = p_{\ell\ell}$ and $p_{rr} = p_{r\ell}$ holds for $i = k$. Hence, p_k is an ec-insertion path in Π_k that starts in a face at side s_{i-1} of e_i and ends in a face adjacent to the copy of w in G_k. We obtain Π by replacing e_k with Γ_{k-1}^- in Π_k, and thus $p_1 + \cdots + p_k$ is an ec-insertion path for v and w in Π. □

Lemma 6. *Let Π' be an arbitrary ec-embedding of B and let p' be a shortest ec-insertion path for v and w in B with respect to Π'. Then $|p'| \geq |p_1 + \cdots + p_k|$.*

Proof. Let G_i, S_i, and s_i be as defined in OptimalEcBlockInserter, and let λ_ℓ^i and λ_r^i be the value of λ_ℓ and λ_r, respectively, after the i-th iteration of the for-loop. For

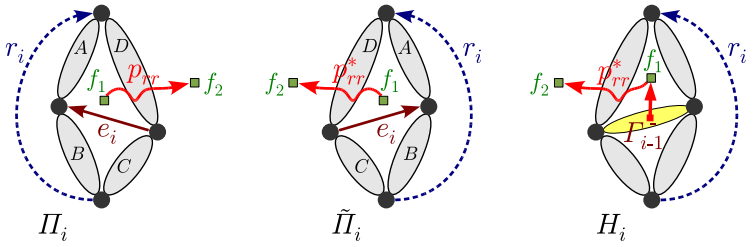


Fig. 6: Proof of Lemma 5; μ_i is an R-node, $R_v = \{f_1\}$, $L_w = \{f_2\}$.

$i = 1, \ldots, k$, we denote with H_i the pertinent graph of μ_i. Observe, that Π' induces embeddings of G_i and S_i. Accordingly, we denote the induced embedding of G_i with Π'_i, and of S_i with Σ'_i.

Since p' is a shortest ec-insertion path, it does not visit a face twice. Therefore, we can subdivide p' into $p' = p'_1 + \cdots + p'_k$ such that p'_i contains exactly the edges of p' that are in G_i, for $1 \leq i \leq k$. This follows from the fact that H_i shares only two vertices with the rest of the graph and p' does not visit a face twice. For $1 \leq i < k$, we denote with $s'_i \in \{\ell, r\}$ the side at which the ec-insertion path $p'_1 + \cdots + p'_i$ leaves H_i in Π'.

We show by induction over i that $\lambda^i_{s'_i} \leq |p'_1 + \cdots + p'_i|$.

$i = 1$. If $k = 1$, then $G_1 = B$ and the proposition follows immediately, so assume $k > 1$. If μ_1 is not an R-node, then $\lambda_{s'_1} = 0$ and the proposition follows immediately. Otherwise, the algorithm also computes the shortest ec-insertion path leaving at side s'_1 in Σ'_1, where the costs of the edges are their traversing costs. Since the traversing costs are independent of the embedding by Lemma 4, we get $\lambda^1_{s'_1} \leq |p'_1|$.

$1 < i < k$. Assume now that $\lambda^j_{s'_j} \leq |p'_1 + \cdots + p'_j|$ for $1 \leq j < i$. We distinguish two cases:

1. **μ_i is a P-node.** In this case, we have $s'_{i-1} = s'_i$, since $p_i + \cdots + p_k$ does not contain an edge of H_{i-1}. This yields
$$\lambda^i_{s'_i} = \lambda^{i-1}_{s'_{i-1}} \leq |p'_1 + \cdots + p'_{i-1}| \leq |p'_1 + \cdots + p'_i|.$$
2. **μ_i is an S- or an R-node.** Observe that p'_i is an ec-insertion path in Π'_i starting in the face at side s'_i of the edge representing v and ending in a face at side $\bar{s}'_{i+1}$ of the edge representing w if $i < k$, or a face adjacent to w otherwise. This implies an ec-insertion path in Σ'_i, where the costs of a skeleton edge are its traversing costs. On the other hand, the algorithm computes a shortest ec-insertion path in Σ'_i, since the traversing costs of a skeleton edge are independent of the embedding by Lemma 4. Thus, we get $\lambda_{s'_i} - \lambda_{s'_{i-1}} \leq |p'_i|$, and hence
$$\lambda_{s'_i} \leq \lambda_{s'_{i-1}} + |p_i| \leq |p'_1 + \cdots + p'_i|.$$

Finally, we get $|p_1 + \cdots + p_k| = \lambda^i_{s'_i} \leq |p'|$ and the lemma holds. □

Theorem 3. *Let $B = (V, E)$ be a block of K and let v and w be two distinct vertices of B. Then, Function* OptimalEcBlockInserter *computes an optimal ec-insertion path for v and w in B in time $O(|E|)$.*

```
procedure OptimalEcInserter(ec-expansion G, vertex v, vertex w)
    Compute the block-vertex tree B of G
    Find the path v, B_1, c_1, ..., B_{k-1}, c_{k-1}, B_k, w from v to w in B.
    for i := 1, ..., k do
        let x_i and y_i be the representatives of v and w in B_i
        p_i := OptimalEcBlockInserter(B_i, x_i, y_i)
    end for
    return p_1 + ... + p_k
end procedure
```

Algorithm 4: Computation of an optimal ec-insertion path.

Proof. The correctness and optimality of the algorithm follows from Lemma 5 and Lemma 6. Constructing the SPQR-tree and embedding the skeleton graphs takes time $O(|E|)$; see [14, 18, 7]. Let $G_i = (V_i, E_i)$ be the graph considered in each iteration of the for-loop. Then, each iteration takes time $O(|E_i|)$, since ShortestEcInsPath takes only time linear in the size of G_i by applying BFS. Moreover, the set E_i consists of some edges E'_i of G plus at most two virtual edges (the representatives of v and w). Thus, $|E_1| + \cdots + |E_k| = O(|E|)$, and hence we get a total running time of $O(|E|)$. □

6.4 Generalization to Connected Graphs

The edge insertion algorithm can easily be generalized to connected graphs by using the same technique as in [16] for the unconstrained case; see Alg. 4. For each block B_i on the path from v to w in the block-vertex tree $\mathcal{B}$ of G, we compute the optimal ec-edge insertion path p_i between the representatives of v and w with a corresponding ec-planar embedding Π_i. Then, we concatenate these ec-edge insertion paths building the optimal ec-edge insertion path for v and w.

The correctness proof in [16] uses induction over the number of blocks on the path from v to w in $\mathcal{B}$. We briefly recall this proof. Let $B_1, \ldots, B_k$ be the blocks on this path and let H_i be the union of the blocks B_1 to B_i. Let Π_i be an embedding of B_i such that p_i is an optimal edge insertion path for the representatives x_i and y_i in B_i with respect to Π_i. Let Ψ_i denote the concatenation $p_1 + \cdots + p_i$.

An embedding Γ_i for H_i with an optimal edge insertion path Ψ_i can be iteratively constructed by combining the embedding Γ_{i-1} for H_{i-1} and the embedding Π_i for block B_i. Both y_{i-1} and x_i denote the same vertex in G and there exist optimal edge insertion paths Ψ_{i-1} for v_1 and y_{i-1} and p_i for x_i and y_i. Therefore there is a face $f \in \Gamma_{i-1}$ that contains y_{i-1} and either v_1 if Ψ_{i-1} is empty or the last edge in Ψ_{i-1}. Similarly, there is a face $f' \in \Pi_i$ that contains x_i and either y_i if p_i is empty or the first edge in p_i. We can directly concatenate the two paths if both faces coincide. This can be achieved by choosing f as the external face of Γ_{i-1} and placing this embedding of H_{i-1} into face f' of Π_i. Then Ψ_i is an optimal ec-insertion path for v_1 and y_i in H_i with respect to Γ_i.

We need to show that—under the presence of embedding constraints—ec-planarity is preserved, i.e., Γ_k is still an ec-planar embedding. The only critical aspect in each step is the selection of f as the external face; but this does not change the clockwise order of the edges around the vertices of G. Furthermore, we ensure in the computation of the ec-edge insertion paths p_i that we do not cross any expansion edges. Hence, we know

that the paths Ψ_{i-1} and p_i do not start or end in a face containing a hub. Therefore, the ec-planarity conditions are still fulfilled and Γ_k is ec-planar.

It is obvious that $p_1 + \cdots + p_k$ is an ec-edge insertion path for v and w with respect to an embedding Π that results from inserting the remaining blocks not contained in H_k (as shown in the proof of Lemma 3) into Γ_k. The length of the computed ec-edge insertion path is obviously minimal, since a shorter path would imply that there exists a shorter path within a block. The block-vertex tree of a graph can be constructed in linear time and the running time of OptimalEcBlockInserter(B_i, x_i, y_i) is linear in the size of the block B_i (Theorem 3), thus yielding linear running time for OptimalEcInserter.

Together with Lemma 1, we obtain the following result:

Theorem 4. *Let $G = (V, E)$ be a graph with embedding constraints C and $e = (v, w) \in E$ such that $G - e$ is ec-planar. Then, we can compute an optimal ec-edge insertion path for v, w in $G - e$ in $O(|V| + |E|)$ time.*

7 Conclusion and Future Work

We introduced a flexible concept of embedding constraints which allows to model a wide range of constraints on the order of edges incident to a vertex. We presented a linear time algorithm for testing ec-planarity, as well as a characterization of all possible ec-embeddings. The latter is in particular important for developing algorithms that optimize over the set of all ec-planar embeddings. We showed that optimal edge insertion can still be performed in linear time when embedding constraints have to be respected. In order to devise practically successful graph drawing algorithms, the following problems should be considered:

- Incorporate the concept of embedding constraints into the planarization approach [1, 15] so that also non-ec-planar graphs can be handled. In particular, algorithms for finding ec-planar subgraphs are required. Notice that this problem can, e.g., be solved in quadratic time using successive ec-planarity testing.
- Solve the so-called *orientation problem* for orthogonal graph drawing, e.g., allow to fix some edges to attach only at the top side of a rectangular vertex.
- In some applications, only a subset of the edges is subject to embedding constraints at a vertex v, i.e., some edges can attach at arbitrary positions. Hence, we wish to extend the concept of embedding constraints for so-called *free edges* that are not contained in the tree T_v.

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